# Chapter 7: Complex numbers

Quadratic equation:

$$a\lambda^2 + b\lambda + c = 0 \ ,$$

with roots

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a} \quad \text{where} \quad D = b^2 - 4ac$$

What if $D < 0$? Define:

$$\mathrm{i}^2 = -1 \quad \text{or equivalently} \quad \mathrm{i} = \sqrt{-1}$$

Solve $\lambda^2 = -3$ by using $\mathrm{i}^2 = -1$: $\lambda^2 = \mathrm{i}^2 \times 3 \quad \text{or} \quad \lambda_{1,2} = \pm\mathrm{i}\sqrt{3}$

So if $D < 0$ write:

$$\lambda_{1,2} = \frac{-b \pm \mathrm{i}\sqrt{-D}}{2a}$$

Solve the equation $\lambda^2 + 2\lambda + 10 = 0$:

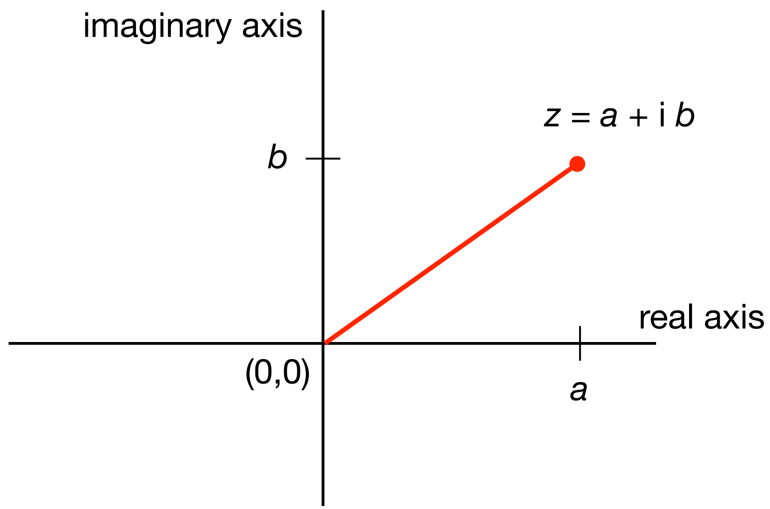
$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4 - 4 \times 10}}{2} = \frac{-2 \pm \sqrt{-36}}{2} = \frac{-2 \pm 6\mathrm{i}}{2}$$

In other words, $\lambda_1 = -1 + 3\mathrm{i}$ and $\lambda_2 = -1 - 3\mathrm{i}$.

A complex number $z$ is written as $z = \alpha + \mathrm{i}\beta$, where $\alpha$ is called the real part and $\mathrm{i}\beta$ is called the imaginary part.

These two solutions are complex conjugates: $z_1 = a + \mathrm{i}b$ and $z_2 = a - \mathrm{i}b$

Argand diagram: complex number as a vector:




Addition of two complex numbers: adding their real parts, and add their imaginary parts.

With $z_1 = 3 + 10\mathrm{i}$ and $z_2 = -5 + 4\mathrm{i}$:

$$z_1 + z_2 = (3 + 10\mathrm{i}) + (-5 + 4\mathrm{i}) = 3 - 5 + 10\mathrm{i} + 4\mathrm{i} = -2 + 14\mathrm{i}\ .$$

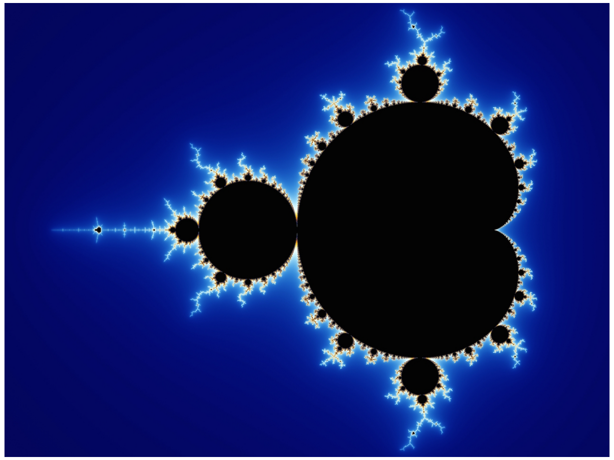
Multiplication works like $(a + bx)(c + dx)$:

$$\begin{aligned} z_1 \times z_2 &= (3 + 10\mathrm{i})(-5 + 4\mathrm{i}) \\ &= 3(-5) + 3 \times 4\mathrm{i} + 10\mathrm{i}(-5) + 10\mathrm{i}4\mathrm{i} \\ &= -15 + 12\mathrm{i} - 50\mathrm{i} + 40\mathrm{i}^2 \\ &= -15 - 38\mathrm{i} - 40 \\ &= -55 - 38\mathrm{i}\ . \end{aligned}$$

Note: $(a + \mathrm{i}b)(a - \mathrm{i}b) = a^2 + b^2$

If $z = a+\mathrm{i}b$, its modulus $|z| = \sqrt{a^2 + b^2}$ (magnitude, length vector).

Hence $z\bar{z} = |z|^2$. (Used for division).



Mandelbrot set: $z_i = z_{i-1}^2 + z_0$, where $z_1 = z_0 = a + b\mathrm{i}$ is a point in the Argand diagram.

Black points remain bounded, colored points keep growing. The color indicates the number of iterations $i = 1, 2, \ldots, n$ required to reach a size of $z_n$.

Start with $z_0 = 0.5$: $0.5, 0.5^2 + 0.5 = 0.75, 0.75^2 + 0.5, \ldots$

# Linear ODEs

$$\begin{cases} \mathrm{d}x/\mathrm{d}t = ax + by \\ \mathrm{d}y/\mathrm{d}t = cx + dy \end{cases} \quad \text{with} \quad \lambda_{1,2} = \frac{\mathrm{tr} \pm \sqrt{D}}{2} \quad \text{and} \quad \begin{cases} (a - \lambda_i)x + by = 0 \\ cx + (d - \lambda_i)y = 0 \end{cases}$$

$$\lambda_{1,2} = \frac{\mathrm{tr} \pm \mathrm{i}\sqrt{-D}}{2} \quad \text{or} \quad \lambda_{1,2} = \alpha \pm \mathrm{i}\beta$$

$$\begin{aligned} \vec{v_1} &= k \begin{pmatrix} -b \\ a - \lambda_1 \end{pmatrix} = k \begin{pmatrix} -b \\ a - (\alpha + \mathrm{i}\beta) \end{pmatrix} \\ &= k \begin{pmatrix} -b \\ a - \alpha \end{pmatrix} - \mathrm{i}k \begin{pmatrix} 0 \\ \beta \end{pmatrix} = k\vec{w_R} - \mathrm{i}k\vec{w_I} \end{aligned}$$

where $\vec{w_R} = \binom{-b}{a-\alpha}$ and $\vec{w_I} = \binom{0}{\beta}$

Similarly

$$\vec{v_2} = k \begin{pmatrix} -b \\ a - \lambda_2 \end{pmatrix} = k \begin{pmatrix} -b \\ a - (\alpha - \mathrm{i}\beta) \end{pmatrix} = k\vec{w_R} + \mathrm{i}k\vec{w_I}$$

General solution:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1(\vec{w_R} - \mathrm{i}\vec{w_I})\mathrm{e}^{(\alpha+\mathrm{i}\beta)t} + C_2(\vec{w_R} + \mathrm{i}\vec{w_R})\mathrm{e}^{(\alpha-\mathrm{i}\beta)t}$$

where the constants $k$ are absorbed into $C_1$ and $C_2$.

Euler's formula:

$$\mathrm{e}^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x \quad \text{or} \quad \mathrm{e}^{-\mathrm{i}x} = \cos x - \mathrm{i}\sin x$$

hence

$$\mathrm{e}^{(\alpha+\mathrm{i}\beta)t} = \mathrm{e}^{\alpha t}\mathrm{e}^{\mathrm{i}\beta t} = \mathrm{e}^{\alpha t}(\cos\beta t + \mathrm{i}\sin\beta t)$$

From

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1(\vec{w_R} - \mathrm{i}\vec{w_I})\mathrm{e}^{(\alpha+\mathrm{i}\beta)t} + C_2(\vec{w_R} + \mathrm{i}\vec{w_R})\mathrm{e}^{(\alpha-\mathrm{i}\beta)t}$$

we obtain

$$\begin{aligned} \left(\begin{smallmatrix} x(t) \\ y(t) \end{smallmatrix}\right) &= C_1(\vec{w_R} - \mathrm{i}\vec{w_I})\mathrm{e}^{\alpha t}(\cos\beta t + \mathrm{i}\sin\beta t) \\ &+ C_2(\vec{w_R} + \mathrm{i}\vec{w_I})\mathrm{e}^{\alpha t}(\cos\beta t - \mathrm{i}\sin\beta t) \\ &= \mathrm{e}^{\alpha t}[C_1(\vec{w_R} - \mathrm{i}\vec{w_I})(\cos\beta t + \mathrm{i}\sin\beta t) \\ &+ C_2(\vec{w_R} + \mathrm{i}\vec{w_I})(\cos\beta t - \mathrm{i}\sin\beta t)]\ . \end{aligned}$$

which dies out whenever $\alpha = \mathrm{tr}/2 < 0$.

Initial condition where $t = 0$, $\mathrm{e}^{\alpha t} = 1$, $\cos \beta t = 1$ and $\mathrm{i} \sin \beta t = 0$,

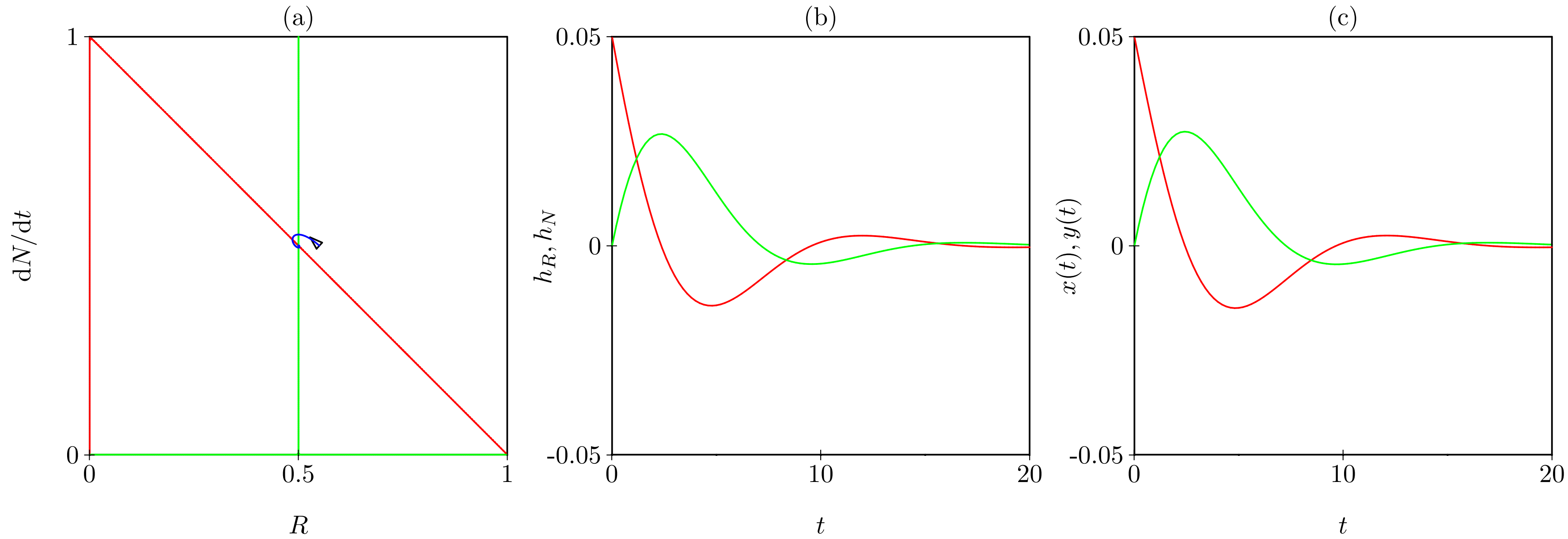
$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = C_1(\vec{w_R} - \mathrm{i}\vec{w_I}) + C_2(\vec{w_R} + \mathrm{i}\vec{w_I})$$
$$= \vec{w_R}(C_1 + C_2) + \mathrm{i}\vec{w_I}(C_2 - C_1) \ , \quad \text{or}$$

$$x(0) = -b(C_1 + C_2) \quad \text{and} \quad y(0) = (a - \alpha)(C_1 + C_2) + \mathrm{i}\beta(C_2 - C_1)$$

from which we solve the complex pair $C_1$ and $C_2$.

Note that $C_1 + C_2$ should be real, whereas $C_2 - C_1$ should be an imaginary number.

# Lotka-Volterra model




$$\frac{\mathrm{d}R}{\mathrm{d}t} = aR - bR^2 - cRN \;, \quad \frac{\mathrm{d}N}{\mathrm{d}t} = dRN - eN$$

With $a = b = c = d = 1, e = 0.5, \bar{R} = 0.5$ and $\bar{N} = 0.5$, and $h_R = 0.05$ and $h_N = 0$

$$J = \begin{pmatrix} -\frac{be}{d} & -\frac{ce}{d} \\ \frac{da-eb}{c} & 0 \end{pmatrix} = \begin{pmatrix} -b\bar{R} & -c\bar{R} \\ d\bar{N} & 0 \end{pmatrix} \, .$$

For $a = b = c = d = 1$ and $e = 0.5$, $\bar{R} = 0.5$ and $\bar{N} = 0.5$, and

$$J = \begin{pmatrix} -0.5 & -0.5 \\ 0.5 & 0 \end{pmatrix} \quad \text{with} \quad D = -0.75$$

implying that

$$\lambda_{1,2} = \frac{\text{tr} \pm \mathrm{i}\sqrt{-D}}{2} \quad \text{or} \quad \lambda_{1,2} = \frac{-0.5 \pm \mathrm{i}\sqrt{0.75}}{2} = -0.25 \pm \mathrm{i}\ 0.43 \ .$$

Hence $\alpha = -0.25$ and $\beta = 0.43$, the nontrivial state is stable, has a return time of $-1/\alpha = 4$, and a wave length proportional to $1/\beta$.

$$\vec{v_1} = \begin{pmatrix} 0.5 \\ -0.25 - \mathrm{i}0.43 \end{pmatrix} \quad \text{and} \quad \vec{v_2} = \begin{pmatrix} 0.5 \\ -0.25 + \mathrm{i}0.43 \end{pmatrix} .$$

$$\binom{x(t)}{y(t)} = \mathrm{e}^{-0.25t} \left[ C_1 \vec{v_1} (\cos 0.43t + \mathrm{i} \sin 0.43t) + C_2 \vec{v_2} (\cos 0.43t - \mathrm{i} \sin 0.43t) \right]$$

$$x(t) = \mathrm{e}^{-0.25t}\ 0.5[(C_1 + C_2) \cos 0.43t + (C_1 - C_2) \mathrm{i} \sin 0.43t]$$

$$y(t) = \ldots$$

Using the initial condition, where $t = 0$, $\mathrm{e}^{-0.25t} = 1$, $\cos 0.43t = 1$, and $\sin 0.43t = 0$, the linearized solution $x(t)$ simplifies into

$$x(0) = 0.05 = 0.5(C_1 + C_2) \quad \text{, and hence} \quad C_1 + C_2 = 0.1 .$$

$y(t)$ simplifies into

$$0 = \mathrm{i}0.43(C_2 - C_1) - 0.25(C_1 + C_2) \quad \leftrightarrow \quad C_2 - C_1 = -\mathrm{i}0.058 \ .$$

We find that $C_1 = 0.05 + \mathrm{i}0.029$ and $C_2 = 0.05 - \mathrm{i}0.029$.

Substituting these constants into $x(t)$ gives

$$\begin{aligned} x(t) &= \mathrm{e}^{-0.25t}0.5[0.1\cos 0.43t + \mathrm{i}^2 0.058\sin 0.43t] \ , \\ &= \mathrm{e}^{-0.25t}[0.05\cos 0.43t - 0.029\sin 0.43t] \end{aligned}$$

and in $y(t)$

$$y(t) = \mathrm{e}^{-0.25t}0.058\sin 0.43t,$$

Both are perfectly real.

# The end