Chapter 8: Modeling chains of ODEs

\[
\frac{dR}{dt} = [r(1 - R/K) - bN]R, \quad \frac{dN}{dt} = [bR - d - cM]N \quad \text{and} \quad \frac{dM}{dt} = [cN - e]M
\]

For odd length chains, \( n = 1 \)
\[\bar{R} = K\]

For even length chains, \( n = 2 \)
\[\bar{R} = \frac{d}{b} \quad \text{and} \quad \bar{N} = \frac{r}{b} \left(1 - \frac{d}{bK}\right) = \frac{r}{b} \left(1 - \frac{1}{R_0}\right)\]
Modeling chains of ODEs

\[
\frac{dR}{dt} = [r(1 - R/K) - bN]R, \quad \frac{dN}{dt} = [bR - d - cM]N \quad \text{and} \quad \frac{dM}{dt} = [cN - e]M
\]

\(n=1\) \(\bar{R} = K\)

\(n=2\) \(\bar{R} = \frac{d}{b}\) and \(\bar{N} = \frac{r}{b} \left(1 - \frac{d}{bK}\right) = \frac{r}{b} \left(1 - \frac{1}{R_0}\right)\)

\(R' = \frac{cr}{be}\)

\(n=3\) \(\bar{N} = \frac{e}{c}, \bar{R} = K \left(1 - \frac{be}{cr}\right)\) and \(\bar{M} = \frac{b\bar{R} - d}{c}\)

For odd chain lengths \(R\) depends on \(K\)
Modeling chains of ODEs

\[
\begin{align*}
\frac{dR}{dt} &= [r(1 - R/K) - bN]R, \\
\frac{dN}{dt} &= [bR - d - cM]N, \\
\frac{dM}{dt} &= [cN - e]M
\end{align*}
\]

![Graphs showing different scenarios](image_url)
Modeling chains with saturated interaction terms

\[
\frac{dR}{dt} = \left[ r\left(1 - \frac{R}{K}\right) - \frac{bN}{h_R + R}\right] R, \quad \frac{dN}{dt} = \left[ \frac{bR}{h_R + R} - d - \frac{cM}{h_N + N}\right] N, \quad \text{and} \quad \frac{dM}{dt} = \left[ \frac{cN}{h_M + M} - e\right] M
\]

**f_R:** \( R \) and \( N \) 

**f_N:** in absence of \( M \) no \( N \) 

**f_M:** no \( M \)

Per capita function always depends on variable itself.

\[
aXY \approx \frac{aXY}{1 + X/k + Y/k} \quad \text{when} \ k \ \text{is large}
\]
8.2 Chains with saturating interacting terms

The results derived in the previous section have to do with the fact that most populations in these models are replicators, which means that their ODEs can be written as \( \frac{d}{dt}x_i = f_i(x) \), where \( x \) is a vector representing the \( n \)-dimensional state of the system. Solving the non-trivial steady state therefore typically involves cancelling the \( x_i = 0 \) solution from its own equation, and subsequently solving \( f_i(x) = 0 \). Since the \( f_i(x) \) terms in Eq. (8.1) correspond to the terms within the square brackets, i.e.,

\[
\begin{align*}
\bar{R} &= r_R \left( \frac{1}{R/K} \right) bN, \\
\bar{N} &= dN \left( \frac{1}{N/K} \right) cM, \\
\bar{M} &= eM.
\end{align*}
\]

we observe that only \( f_R \) depends on itself, i.e., on \( R \). Because \( f_N \) and \( f_M \) are independent of \( N \) and \( M \), respectively, their steady states are necessarily solved from another equation, i.e., \( \bar{N} \) from \( f_M = 0 \), then \( \bar{R} \) from \( f_R = 0 \), and finally \( \bar{M} \) from \( f_N = 0 \).
SEIR model:

\[
\begin{align*}
\frac{dS}{dt} &= s - dS - \beta SI, \\
\frac{dE}{dt} &= \beta SI - (d + \gamma)E, \\
\frac{dI}{dt} &= \gamma E - (\delta + r)I \quad \text{and} \\
\frac{dR}{dt} &= rI - dR
\end{align*}
\]

\[
\begin{align*}
\bar{R} &= \frac{r}{d} \bar{I}, \\
\bar{I} &= \frac{\gamma}{\delta + r} \bar{E}, \\
\bar{S} &= \frac{(d + \gamma)(\delta + r)}{\gamma \beta}, \\
\bar{E} &= \frac{s}{d + \gamma} - \frac{d(\delta + r)}{\gamma \beta}
\end{align*}
\]

\(\bar{R}\) and \(\bar{I}\) are proportional to previous level
Cascade of cell divisions

\[
\frac{dN_0}{dt} = s - (p + d)N_0, \quad \frac{dN_i}{dt} = 2pN_{i-1} - (p + d)N_i \quad \text{and} \quad \frac{dN_n}{dt} = 2pN_{n-1} - dN_n,
\]

\[
\bar{N}_0 = \frac{s}{p + d}, \quad \bar{N}_i = \frac{2p}{p + d} \bar{N}_{i-1} \quad \text{and} \quad \bar{N}_n = \frac{2p}{d} \bar{N}_{n-1}
\]

\[
J = \begin{pmatrix}
-p-d & 0 & 0 & 0 & \cdots & 0 \\
2p & -(p+d) & 0 & 0 & \cdots & 0 \\
0 & 2p & -(p+d) & 0 & \cdots & 0 \\
& & \vdots & & \ddots & \ddots \\
0 & \cdots & 0 & \cdots & 0 & 2p & -d
\end{pmatrix}
\]

\[
(J_{00} - \lambda)(J_{11} - \lambda)(J_{22} - \lambda) \ldots (J_{nn} - \lambda) = 0
\]
Cascade of cell divisions

\[
\frac{dN_0}{dt} = s - (p + d)N_0, \quad \frac{dN_i}{dt} = 2pN_{i-1} - (p + d)N_i \quad \text{and} \quad \frac{dN_n}{dt} = 2pN_{n-1} - dN_n,
\]

\[
\bar{N}_0 = \frac{s}{p + d}, \quad \bar{N}_i = \frac{2p}{p + d} \bar{N}_{i-1} \quad \text{and} \quad \bar{N}_n = \frac{2p}{d} \bar{N}_{n-1}
\]

\[
\bar{N}_0 = \frac{s}{p + d}, \quad \bar{N}_i = \frac{2^i p^i s}{(p + d)^{i+1}} \quad \text{and} \quad \bar{N}_n = \frac{s}{d} \left( \frac{2p}{p + d} \right)^n
\]

\[
\frac{dQ}{dt} = -aQ - dQQ + d \sum f_i N_i \quad \text{and} \quad s = aQ
\]
Kinetic proofreading

Michaelis Menten:

\[ F + L \overset{k_1}{\underset{k_{-1}}{\rightleftharpoons}} C \quad \text{or} \quad \frac{dC}{dt} = k_1 FL - k_{-1} C \quad \text{with} \quad F = R - C \quad \text{gives} \quad C = \frac{RL}{K_m + L} \]

Kinetic proofreading:

\[ F + L \overset{k_1}{\underset{k_{-1}}{\rightleftharpoons}} C_0 , \quad C_{i-1} \overset{k_2}{\rightarrow} C_i \quad \text{and} \quad C_i \overset{k_{-1}}{\rightarrow} F \]

\[ \frac{dC_0}{dt} = k_1 FL - (k_{-1} + k_2)C_0 , \quad \frac{dC_i}{dt} = k_2 C_{i-1} - (k_{-1} + k_2)C_i \quad \text{and} \quad \frac{dC_n}{dt} = k_2 C_{n-1} - k_{-1} C_n \]

with \( F = R - \sum_i^n C_i \) gives \( \bar{C}_n = \frac{RL}{K_m + L} \left( \frac{k_2}{k_{-1} + k_2} \right)^n \)
8.3 Other famous chain models

\[ F + L \overset{k_1}{\underset{k_{-1}}{\rightleftharpoons}} C_0, \quad C_{i-1} \overset{k_2}{\longrightarrow} C_i \quad \text{and} \quad C_i \overset{k_{-1}}{\longrightarrow} F \quad \text{where} \quad F = R - C \]

\[ \frac{dC_0}{dt} = k_1 FL - (k_{-1} + k_2)C_0, \quad \frac{dC_i}{dt} = k_2 C_{i-1} - (k_{-1} + k_2)C_i \quad \text{and} \quad \frac{dC_n}{dt} = k_2 C_{n-1} - k_{-1}C_n \]

\[ \bar{C}_n = \frac{RL}{K_m + L} \left( \frac{k_2}{k_{-1} + k_2} \right)^n \]