Chapter 6: $R_0$

SIR model:

$$\frac{dS}{dt} = s - dS - \beta SI,$$
$$\frac{dI}{dt} = \beta SI - (\delta + r)I,$$
$$\frac{dR}{dt} = rI - dR,$$

Virulence: $v = \delta - d$

$$\bar{S} = \frac{\delta + r}{\beta}, \quad \bar{I} = \frac{s}{\delta + r} - \frac{d}{\beta}, \quad \text{and} \quad \bar{R} = \frac{rs}{d(\delta + r)} - \frac{r}{\beta},$$

$$R_0 = \beta \bar{S} \frac{1}{\delta + r} = \frac{s}{d} \frac{\beta}{\delta + r}$$
$R_0$ can also be computed from Jacobian

**SIR model:**

\[
\frac{dS}{dt} = s - dS - \beta SI, \quad \frac{dI}{dt} = \beta SI - (\delta + r)I, \quad \text{and} \quad \frac{dR}{dt} = rI - dR,
\]

**SI model (l=0):**

\[
R_0 = \beta \bar{S} \frac{1}{\delta + r} = s, \quad \frac{\beta}{d} \frac{\beta}{\delta + r}
\]

\[
\lambda_1 < 0 \text{ same as } R_0 > 1 \quad \lambda_1 = \beta \bar{S} - \delta - r \text{ and } \lambda_2 = -d.
\]
R_0 is not equal to the growth/expansion rate of the epidemic

SIR model:

\[
\frac{dS}{dt} = s - dS - \beta SI , \quad \frac{dI}{dt} = \beta SI - (\delta + r)I , \quad \text{and} \quad \frac{dR}{dt} = rI - dR , \quad R_0 = \beta \bar{S} \frac{1}{\delta + r} = \frac{s}{d} \frac{\beta}{\delta + r}
\]

Initial expansion rate (r_0 > 0):

\[
\frac{dI}{dt} = \beta \bar{S}I - (\delta + r)I = \left( \frac{\beta s}{d} - \delta - r \right) I = r_0 I , \quad \lambda_1 = \beta \bar{S} - \delta - r
\]
$R_0$ is not equal to the growth/expansion rate of the epidemic

Initial expansion rate $(r_0 > 0)$:

$$\frac{dI}{dt} = \beta \tilde{S} I - (\delta + r) I = \left( \frac{\beta s}{d} - \delta - r \right) I = r_0 I, \quad R_0 = \beta \tilde{S} \frac{1}{\delta + r} = \frac{s}{d} \frac{\beta}{\delta + r}$$

Define length of the infectious period, $L = \frac{1}{\delta + r}$ to see that

$$R_0 = \beta \tilde{S} L \quad \text{and} \quad r_0 = \beta \tilde{S} - \frac{1}{L} \quad \text{such that} \quad \beta \tilde{S} = r_0 + \frac{1}{L} \quad \text{and} \quad R_0 = r_0 L + 1$$

From the latter: $r_0 = \frac{R_0 - 1}{L} = (R_0 - 1)(\delta + r)$
Frequency dependent infections

SIR model:

\[
\frac{dS}{dt} = s - dS - \beta SI , \quad \frac{dI}{dt} = \beta SI - (\delta + r)I , \quad \text{and} \quad \frac{dR}{dt} = rI - dR ,
\]

\[
R_0 = \frac{s \beta}{d \delta + r}
\]

Frequency dependent infection in SIR model: \( N = S + I + R \)

\[
\frac{dS}{dt} = s - dS - \frac{\beta SI}{N} , \quad \frac{dI}{dt} = \frac{\beta SI}{N} - (\delta + r)I , \quad \text{and} \quad \frac{dR}{dt} = rI - dR ,
\]

\[
R_0 = \frac{\beta}{\delta + r}
\]
6.2 The SEIR model

\[ \frac{dS}{dt} = s - dS - \beta SI, \quad \frac{dE}{dt} = \beta SI - (\gamma + d)E, \quad \frac{dI}{dt} = \gamma E - (\delta + r)I, \quad \frac{dR}{dt} = rI - dR, \]

where the exposed individuals become infectious at a rate \( \frac{d}{\gamma + d} \), \( \frac{d}{\delta + r} \), and this will occur over an infectious period of \( \frac{1}{\gamma} \), that are not yet infectious, one obtains \( \bar{I} \).

Summarizing, for multi-stage models is the "next generation method" devised by Diekmann et al. 1990, and involves the definition of a matrix \( R_0 \) which can only be positive when \( R_0 > 1 \) is the same as \( \bar{I} > 0 \).
SEIR model \[ R_0 = \frac{s}{d} \frac{\beta}{\delta + r} \frac{\gamma}{\gamma + d}. \]

\[
\frac{dS}{dt} = s - dS - \beta SI , \quad \frac{dE}{dt} = \beta SI - (\gamma + d)E , \quad \frac{dI}{dt} = \gamma E - (\delta + r)I , \quad \frac{dR}{dt} = rI - dR ,
\]

\[
J = \begin{pmatrix}
\frac{\partial E}{\partial E} & \frac{\partial E}{\partial I} \\
\frac{\partial E}{\partial I} & \frac{\partial I}{\partial I}
\end{pmatrix} = \begin{pmatrix}
-(\gamma + d) & \beta \bar{S} \\
\gamma & -(\delta + r)
\end{pmatrix}
\]

\[
\text{tr} < 0 \text{ and } \text{det} = (\gamma + d)(\delta + r) - \gamma \beta \bar{S}
\]

\[
\text{det } J < 0 \quad \text{or} \quad \frac{d}{s} \frac{\gamma + d}{\gamma} \frac{\delta + r}{\beta} - 1 < 0 \iff \frac{1}{R_0} < 1 \iff R_0 > 1
\]
Fitness in consumer-resource models

\[
\frac{dR}{dt} = bR(1 - R/k) - dR - aRN \quad \text{and} \quad \frac{dN}{dt} = caRN - \delta N
\]

\[K = k(1 - d/b) \quad R_{0R} = b/d. \quad R_{0N} = caK/\delta\]

\[
\frac{dR}{dt} = bR(1 - R/k) - dR - aRN \quad \text{and} \quad \frac{dN}{dt} = \frac{\beta RN}{h + R} - \delta N
\]

\[R_0 = \frac{\beta K}{\delta(h+K)} \quad \text{or} \quad R_0 = \frac{\beta}{\delta}
\]

\[K = k(1 - d/b)\]