Question 1. Basics and nullclines (10 points)

A simple model for a symbiotic relationship between two populations is the following

$$\frac{dN_1}{dt} = N_1 \left[\frac{b_1 N_2}{h_1 + N_2} - d_1 \left(1 + \frac{N_1}{k_1} \right) \right] \quad \text{and} \quad \frac{dN_2}{dt} = N_2 \left[\frac{b_2 N_1}{h_2 + N_1} - d_2 \left(1 + \frac{N_2}{k_2} \right) \right] ,$$

where the b_i parameters are birth rates, and the d_i parameters are death rates.

a. give a simple interpretation of the h_1 and the k_1 parameter

h_1 is the density of N_2 where N_1 individuals achieve half their maximal birth rate, and k_1 is the density of N_1 where the death rate doubles.

b. Do these populations suffer from intraspecific or interspecific competition or both?

Just intraspecific via the density dependent death rate.

c. sketch the nullclines for a situation where they intersect, and provide the vector field.

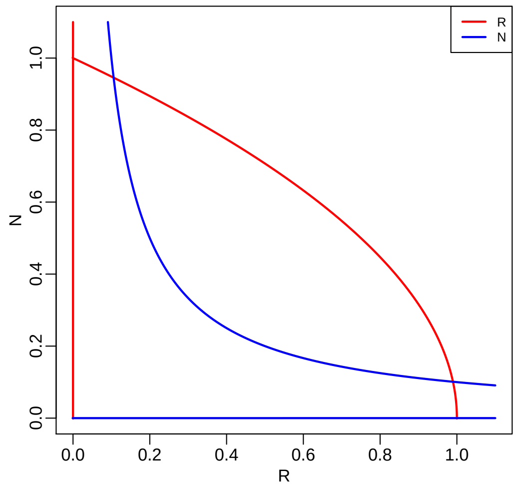
See the answer to Question 10.4a in the book.

Question 2. Stability (10 points)

A simple model for a system where two predators are required to kill a prey would be

$$\frac{dR}{dt} = rR(1 - R/K) - aRN^2 \quad \text{and} \quad \frac{dN}{dt} = caRN^2 - dN ,$$

and this model has the following nullclines



a. Add the vector field.

Start in the upper right-hand corner with (-,+) to find out that $dN/dt < 0$ under the blue line and $dR/dt > 0$ under the red line.

b. Try to determine the stability of all steady states and provide the graphical Jacobian of the two non-trivial steady states.

(0,0) is unstable because of the horizontal arrow point outwards, (1,0) is stable because the vectors point inwards. (0.98,0.1) is a saddle because of the 'vertical' direction point outwards. (0.15,0.9) needs a Jacobian, which is $J = \begin{pmatrix} - & - \\ + & + \end{pmatrix}$. Because both the trace and the determinant have an unknown sign its stability cannot be determined. Note that the saddle point has the same Jacobian!

Question 3. Make a model: 10 points

Consider a fungal species growing as a population of single cells on a limiting amount of material. As a consequence, the division rate declines with the population density. The cells occur as one of 3 types, i.e., '+', '−' and '±' cells, (call them 'x', 'y', and 'z' cells in your model) that equally compete for the material, and with equal birth and death rates. When cells divide the two daughters both inherit the type of their mother. When + and − cells meet they can fuse into a single ± cell. The ± cells occasionally split into a + and a − cell. Thus all 3 types are typically maintained in the population.

Write a natural model for a population of this species.
$dx/dt = sz + bx(1 - T/k) - dx - fxy$, $dy/dt = sz + by(1 - T/k) - dy - fxy$, $dz/dt = fxy + bz(1 - T/k) - dz - sz$, and $T = x + y + z$.

Question 4. Extend a model: 10 points

We have modeled bacteria growing in a chemostat with the following model

$$\frac{dR}{dt} = s - wR - \frac{aRN}{h+R} \quad \text{and} \quad \frac{dN}{dt} = \frac{caRN}{h+R} - (w+d)N \ ,$$

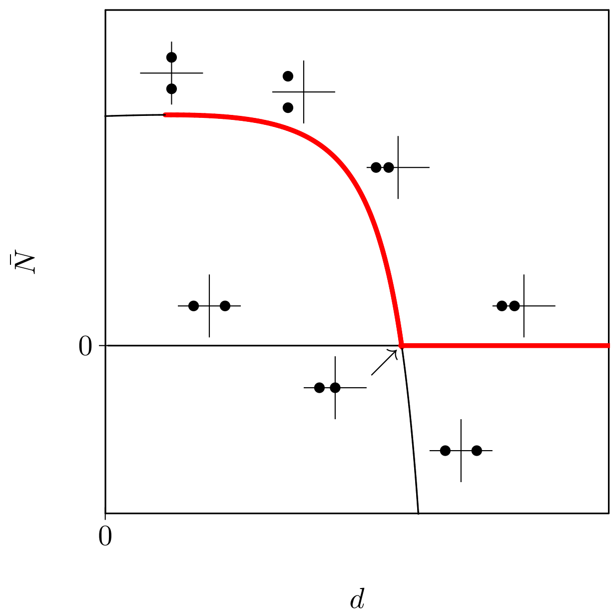
where w is the 'wash out' rate from the chemostat, and the bacteria, N, consume resource, R, using a Monod saturation function. In some species the bacteria produce a toxin when the bacterial densities become high, and this toxin increases their own death rate. Extend this model with such a toxin.
$dT/dt = pN\frac{N}{k+N} - wT$ and $dN/dt = \frac{caRN}{h+R} - (w + d + xT)N$.

Question 5. Transcritical bifurcation: 10 points

In the book we have changed the predator death death rate, d, in a saturated predator/prey-model,

$$\frac{dR}{dt} = rR(1 - R/K) - \frac{bR^2N}{h^2+R^2} \quad \text{and} \quad \frac{dN}{dt} = \frac{cbNR^2}{h^2+R^2} - dN \ ,$$

to obtain the following bifurcation diagram revealing a Transcritical bifurcation



Interpret this bifurcation diagram in less than 200 words. For instance, use the terms 'Transcritical bifurcation', 'Hopf bifurcation', 'largest eigenvalue', 'stable node', 'saddle point' and 'stable spiral point'. It is also fine to provide a sketch illustrating your story.
The curved line depicts the non-trivial steady state of this model. On the left we see a complex pair of eigenvalues of an unstable spiral point acquiring a negative real part at a Hopf bifurcation, and hence forming a stable spiral point. This spiral point becomes a stable node, which collapses in a Transcritical bifurcation with the horizontal line depicting the $(\bar{R} = K, \bar{N} = 0)$ state of this model. This carrying capacity state is a saddle below the Transcritical bifurcation and becomes a stable node after it when the largest eigenvalue becomes negative.