Exam Biological Modeling (B-B2THEC05 & B-MBIMOD) 13 January 2022, Educ-Gamma

- This exam is composed of 5 questions (pages are double-sided).
- You can work on this exam for 2.5 hours (13:30–16:00h). The minimal residence time is half an hour (to allow for late comers).
- Provide your answers on the answering-sheets we provide and **write your name and student-number on each sheet**.
- Have a photo-id on your table.
- This is a closed-book exam: you can only use the sheet with formulas and the math-reader.
- Electronic devices (including smartwatches) should be switched off and stored.
- Raise your hand if you need additional sheets of paper, when something is unclear, or if you need to visit a restroom.
- Questions can be answered in English or in Dutch.

We wish you a lot of success!

Rob de Boer

Scratch paper

Question 1. Sewage plant (10 points)
Consider a model for nutrients, N, that are pumped in a sewage-water treatment plant (rioolwaterzuiveringsinstallatie) at a rate s (kg per hour). The nutrients are being consumed by micro-organisms, M (kg), that multiply upon the consumption of nutrients, and are filtered out at rate f. The water with nutrients exits the treatment plant at a rate e. We write

$$\frac{\mathrm{d}N}{\mathrm{d}t} = s - aNM - eN \quad \text{and} \quad \frac{\mathrm{d}M}{\mathrm{d}t} = caNM - fM \ .$$

a. Give the dimension and biological interpretation of the parameters a and c.
c is dimensionless because both N and M are measured in kg. a is per h per kg to make the whole aNM term kg/h.

b. What is the expected amount of nutrients in absence of the micro-organisms?
s/e kg

c. What is the return time of the model in absence of the micro-organisms?
$\partial_N \mathrm{d}N/\mathrm{d}t = -e$ and hence $T_R = 1/e$.

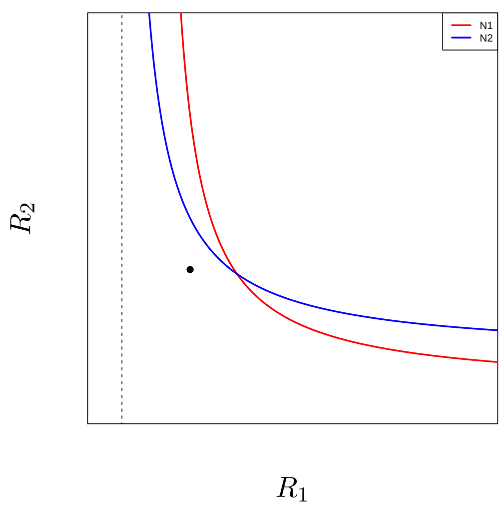
d. What is the expected concentration of nutrients when the micro-organisms are present?
$\mathrm{d}M/\mathrm{d}t = 0$ gives $N = f/(ca)$

e. Draw the phase space such that there is at least one non-trivial equilibrium and indicate the vector field.
See Fig 5.1a in the book

f. Determine the stability of all steady states
We need a graphical Jacobian for the non-trivial steady state $J = \begin{pmatrix} - & - \\ + & 0 \end{pmatrix}$. For the carrying capacity of the nutrients one can see that it is a saddle because $\mathrm{d}M/\mathrm{d}t > 0$.

g. The owner of the treatment facility advertises that the plant is an automatic homeostatic system because the steady state of the micro-organisms 'automatically' increases in periods where more nutrients are pumped into the system. Is that true? Explain in maximally 40 words.
Yes, if s increases N stays at $f/(ca)$ and hence the enrichment ends up in $\bar{M}$. One can also see that the $\mathrm{d}N/\mathrm{d}t = 0$ nullcline moves to the right and upwards.

Question 2. Tilman diagram (10 points)
Consider the following Tilman diagram for two consumers, N_1 and N_2, using two resources:



where the bullet denotes the steady state of the resources in the absence of consumers, and the dashed line is the vertical asymptote of the blue line. The resource equations are

$$\frac{\mathrm{d}R_1}{\mathrm{d}t} = s_1 - d_1R_1 - \sum_j c_{j1}N_jR_1 \quad \text{and} \quad \frac{\mathrm{d}R_2}{\mathrm{d}t} = s_2 - d_2R_2 - \sum_j c_{j2}N_jR_2 \ .$$

a. What are the coordinates of the bullet?

$R_1 = s_1/d_1$ and $R_2 = s_2/d_2$

b. What is the biological interpretation of the vertical dashed line?

This indicates R^*_{21} the critical R_1 density that N_2 needs to start growing.

c. Do you expect that the two consumers can coexist?

No: the intersection point of their nullclines lies above the maximal resource densities.

Question 3. Bacteria and neutrophils (10 points)

Bacteria, B, grow exponentially in a local environment. This area is scavenged by neutrophils, N, killing the bacteria, and having a saturated functional response. The neutrophils have a basic immigration rate, i, into the enviroment, and increase their immigration rate by chemotaxis if there are bacteria in the local environment. We write the following model:

$$\frac{\mathrm{d}B}{\mathrm{d}t} = rB - \frac{kBN}{h_1 + B} \quad \text{and} \quad \frac{\mathrm{d}N}{\mathrm{d}t} = i + \frac{mB}{h_2 + B} - dN \ ,$$

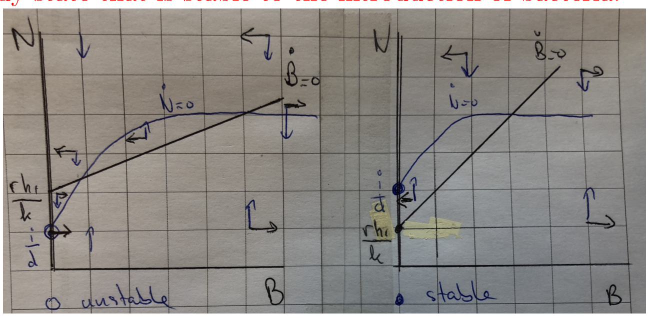
where $1/d$ is the residence time of the neutrophils in the environment.

a. What is the interpretation of the parameters m and h_2?

m is the maximum extra influx of the neutrophils and h_2 is the bacterial density at which the extra influx is $m/2$.

b. Sketch the nullclines for all possible situations where they intersect and add the vector field.

The bacterial nullcline is $N = (r/k)(h_1 + B))$ which is a straight line with a positive slope starting at $N = rh_1/k$. The neutrophil nullcline is $N = (i/d) + (m/d)B/(h_2 + B)$ which is an increasing saturation function starting at $N = i/d$. When $rh_1/k > i/d$ we obtain the phase plane on the left, with an infected steady state that is unstable to the introduction of bacteria. Otherwise we obtain the phase plane on the right, with an infected steady state that is stable to the introduction of bacteria:



c. What happens at very high bacterial concentrations (this can be seen from the equations and from your phase planes)?

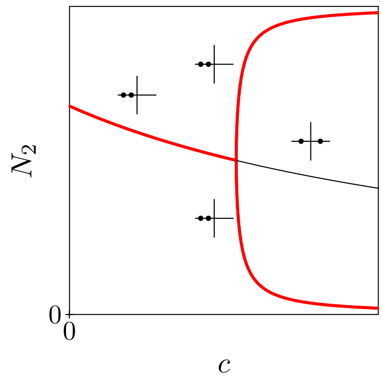
The phase plane reveals an explosion region where the bacteria grow exponentially and the neutrophils remain at $N = (i + m)/d$. In the model we see that for $b \to \infty$ we find $\bar{N} = (i + m)/d$ and hence $\mathrm{d}B/\mathrm{d}t = rB - k(i+m)/d$. At large B this is always positive meaning that the bacteria grow in an unlimited manner.

Question 4. Bifurcations (5 points)

For the simple competition model

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = a + rN_1(1 - N_1 - cN_2) \quad \text{and} \quad \frac{\mathrm{d}N_2}{\mathrm{d}t} = a + rN_2(1 - N_2 - cN_1) \ ,$$

we obtained the following Pitchfork bifurcation diagram:



where the insets are Argand diagrams indicating the values of the two eigenvalues (they are all real in this example).

a. What is the biological interpretation of the a parameter?

an immigration or source term.

b. Sketch the nullclines for a value of c left of the bifurcation point and for a value of c right of the bifurcation point.

See Fig 11.5 in the book.

c. What are the signs of the eigenvalues in the right-most Argand diagram?

One is negative and the other is positive

Question 5. Make a model (10 points)

Write a natural model for an epidemic with two viral strains that both have an $R_0 > 1$. The first one is more virulent than the second, and the second one is more infectious than the first (like corona delta and omicron). Assume that a (small) fraction of the people infected with one of these viruses dies, that most people recover, and are immune to both viruses for a few months (i.e., there are no super infections). Assume for simplicity that recovered people loosing their immunity become fully susceptible to both viruses again. Write a natural SIR model for susceptible people, people infected with virus1, people infected with virus2, recovered people, and dead bodies, and indicate how the parameters should differ between the strains to exactly match this story (i.e., only allow for different parameters if this is truly necessary to match the story).

$\mathrm{d}S/\mathrm{d}t = -\beta_1 I_1 S - \beta_2 I_2 S + wR$, $\mathrm{d}I_1/\mathrm{d}t = \beta_1 I_1 S - d_1 I_1 - rI_1$, $\mathrm{d}I_2/\mathrm{d}t = \beta_2 I_2 S - d_2 I_1 - rI_2$, $\mathrm{d}R/\mathrm{d}t = r(I_1 + I_2) - wR$ and $\mathrm{d}D/\mathrm{d}t = d_1 I_1 + d_2 I_2$, where $d_1 > d_2$ and $\beta_2 > \beta_1$.

Answering sheet

Name: **Student number:**