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Exam Biological Modeling (B-B2THEC05 & B-MBIMOD) 15 October 2020, Theatron

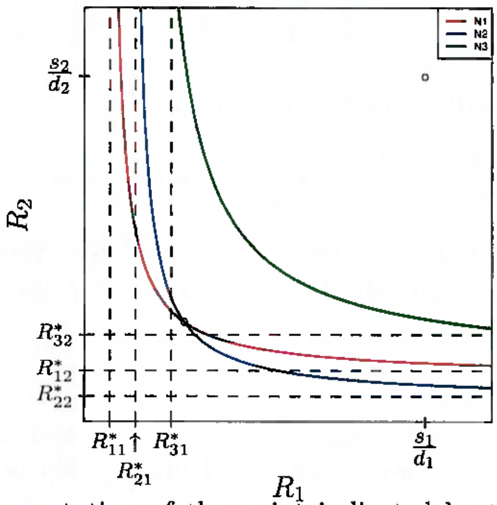
Question 1. Jacobian (4 points)
Consider the Lotka-Volterra competition model for two species:

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = r_1 N_1 - a_{11} N_1^2 - a_{12} N_1 N_2 \quad \text{and} \quad \frac{\mathrm{d}N_2}{\mathrm{d}t} = r_2 N_2 - a_{22} N_2^2 - a_{21} N_2 N_1 \ .$$

Assume that the nullclines intersect in a non-trivial steady state $(\bar{N}_1, \bar{N}_2)$ and write the full Jacobi matrix for that steady state (without filling in the values of $\bar{N}_1$ and $\bar{N}_2$).

Question 2. Tilman diagram (8 points)
In the chapter on competition we discussed the following Tilman diagram for three consumers, N_1, N_2, N_3, using two resources R_1 and R_2. The resource equations are

$$\frac{\mathrm{d}R_1}{\mathrm{d}t} = s_1 - d_1 R_1 - \sum_j c_{j1} N_j R_1 \quad \text{and} \quad \frac{\mathrm{d}R_2}{\mathrm{d}t} = s_2 - d_2 R_2 - \sum_j c_{j2} N_j R_2$$



a. What is the biological interpretation of the point indicated by the open circle at $R_1 = \frac{s_1}{d_1}$ and $R_2 = \frac{s_2}{d_2}$?
b. From the labels on the axes we read that $R^*_{11} < R^*_{21}$ and that $R^*_{22} < R^*_{12}$. Explain in your own words what this means in biological terms (no more than 50 words).
c. Can the third consumer, N_3, invade when N_1 and N_2 are present at the steady state depicted by the intersection point of the nullclines of the first two consumers (see the lower circle)? Explain your answer shortly.

Question 3. Phase space (4 points)
Consider the following model:

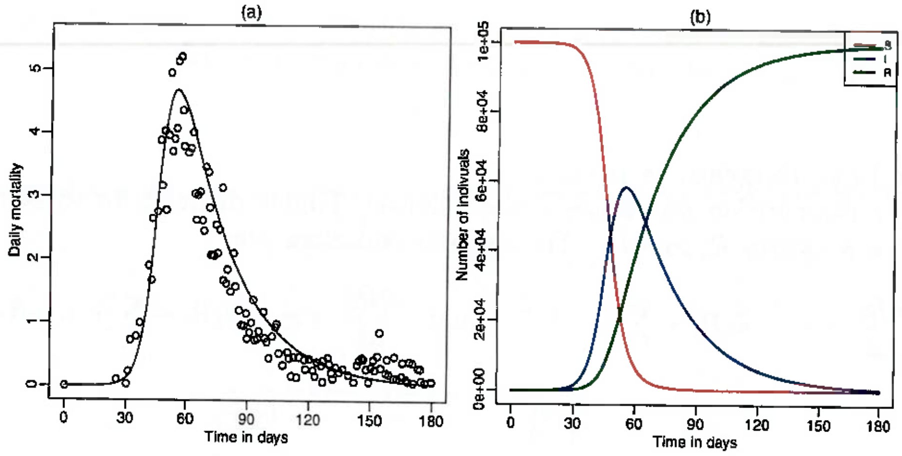
$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = N_1 \left[\frac{\beta_1 N_2}{h + N_2} - \delta_1(1 + \alpha_1 N_1)\right] \quad \text{and} \quad \frac{\mathrm{d}N_2}{\mathrm{d}t} = N_2 \left[\frac{\beta_2 N_1}{h + N_1} - \delta_2(1 + \alpha_2 N_2)\right] \ .$$

a. Sketch the nullclines.
b. How would you define the interaction between the two populations?

Question 4. Make a model (6 points)
Write a natural model for a replicating population of bacteria that can become infected by non-lytic bacteriophages. After infection, non-lytic phages reproduce with the bacteria, i.e., when infected bacteria divide both daughter cells will be infected, and free phages are only released into medium when infected bacteria die. The bacteria are competing for nutrients in their growth medium that is continuously refreshed. The bacteria and the phages are not 'refreshed': due to a 'reflecting' filter

they are not diluted when the medium is refreshed. Write a natural model for the resource, uninfected and infected bacteria, and the bacteriophages (4 ODEs).

Question 5. COVID-19 (10 points)
A recent paper describes an uncontrolled COVID-19 epidemic in a big city in Brazil by documenting the daily mortality per 10^5 inhabitants. The symbols in Panel (a) depict their data, where day zero is the first of March 2020 and the last day is September 1, 2020:



This epidemic can be described with a classic SIR model,

$$\frac{dS}{dt} = -\beta S\frac{I}{N}\ , \quad \frac{dI}{dt} = \beta S\frac{I}{N} - (d+r)I \quad \text{and} \quad \frac{dR}{dt} = rI\ ,$$

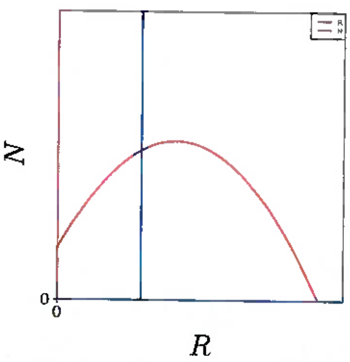
with susceptible individuals, S, infected individuals, I, and recovered individuals, R. Here $N = S+I+R$ is the total number of individuals, β the infection rate, r the recovery rate, and d the death rate due to COVID-19.

a. What is the R_0 of this infection?
b. Which fraction of the infected individuals die?
c. Panel (a) suggests that the epidemic peaks after two months, and subsequently declines. At what fraction of the susceptibles, S/N, would the model predict this peak in the number of infected individuals to occur?
d. What would be the differential equation describing the increase in the number of dead bodies?

This model was fitted to the mortality data to estimate the parameters β, d and r, and the best fit is shown by the line in Panel (a), with the corresponding behavior of the model in Panel (b).

e. Which variable or term of the model was fitted to the data to estimate these three parameters?
f. Explain in a few sentences how such a fitting procedure works: use the term SSR (for summed squared residuals).

Question 6 Graphical Jacobian (4 points)



The phase diagram on the left depicts the nullclines of the classic Monod-saturated predator-prey model. The red line is the nullcline of the prey, R, and the blue line that of the predator N.

a. Determine the stability of the non-trivial steady state using the graphical Jacobian method. Explain how your obtained your result.
b. What kind of behavior of the model do you expect for this phase plane? Sketch a trajectory starting with a few predators, introduced in a prey population at carrying capacity.

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①

$$\begin{array}{c|cc} & N_1 & N_2 \\ \dot{N}_1 & r_1 - 2a_{11}\bar{N}_1 - a_{12}\bar{N}_2 & -a_{12}\bar{N}_1 \\ \dot{N}_2 & -a_{21}\bar{N}_2 & r_2 - 2a_{22}\bar{N}_2 - a_{21}\bar{N}_1 \end{array}$$

2 points for diagonal — 2 points

2 points for off-diagonal — 2 points

② a. This is the carrying capacity of the resources — 2 points

b. $R_{11}^* < R_{21}^*$ means that N_1 needs less of R_1 than N_2 does. $R_{22}^* < R_{12}^*$ means that N_2 needs less of R_2 than N_1. — 4 points

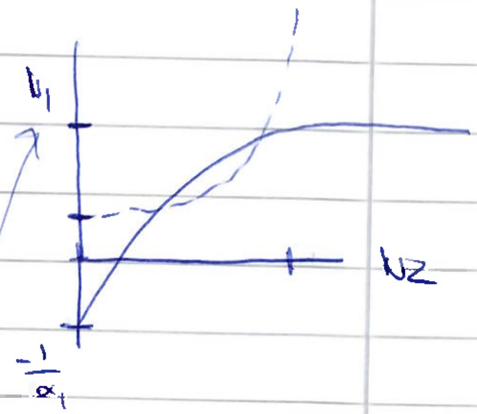
c. No, that point is located below the $\dot{N}_3 = 0$ nullcline. — 2 points

③ a. $\dot{N}_1 = 0$: $N_1 = 0$ or $\dfrac{\beta_1 N_2}{h + N_2} - \delta_1 = \delta_1 \alpha_1 N_1$

$$N_1 = \frac{1}{\alpha_1}\left[\frac{\beta_1/\delta_1 \; N_2}{h + N_2} - 1\right]$$

$N_2 = 0 \rightarrow N_1 = -\dfrac{1}{\alpha_1}$

$N_2 \rightarrow \infty \rightarrow N_1 \rightarrow \dfrac{\beta_1/\delta_1 - 1}{\alpha_1}$



3 points

$\dot{N}_2 = 0$ is just the other way aroun ...

b. symbiotic — 1 "

4
$$\frac{dB}{dt} = \overset{1}{\boxed{aBR}} - \overset{2}{\beta BV} - dB$$

$$\frac{dR}{dt} = \overset{3}{s - d_R R} - \boxed{a(B+1)R}$$

$$\frac{dI}{dt} = \beta BV + \underset{4}{\boxed{aIR}} - dI$$

$$\frac{dV}{dt} = \underset{5}{dI} - \underset{6}{cV}$$

6 points

5 a. $R_0 = \frac{\beta}{d+r}$ 1 points

b. fraction dying is $\frac{d}{d+r}$ 1 "

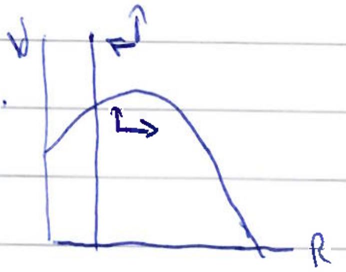
c. $\beta \frac{S}{N} = d+r \Leftrightarrow \frac{S}{N} = \frac{d+r}{\beta} = \frac{1}{R_0}$ 2 "

d. $dD/dt = dI$ 2 "

e. dI 2 "

f. From an initial guess the parameters are changed such that the SSR decreases. One stops when the SSR stops decreasing 2p.

6 a.



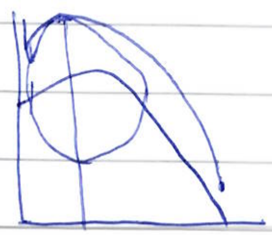
$$J = \begin{array}{c|cc} & R & N \\ \hline \dot{R} & + & - \\ \dot{N} & + & 0 \end{array}$$

$tr > 0$

not stable

3 points

b.



trajectory approaching and dips.

1 point