NON-LINEAR DYNAMICAL SYSTEMS

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NON-LINEAR DYNAMICAL SYSTEMS

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<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.4</td>
<td>General solution of a system with complex eigenvalues</td>
<td>37</td>
</tr>
<tr>
<td>6.4.1</td>
<td>Canonical or Jordan normal form</td>
<td>41</td>
</tr>
<tr>
<td>6.5</td>
<td>Equilibria types for complex eigenvalues</td>
<td>41</td>
</tr>
<tr>
<td>6.6</td>
<td>Stability of equilibrium</td>
<td>42</td>
</tr>
<tr>
<td>6.7</td>
<td>Express method for finding type of equilibrium</td>
<td>42</td>
</tr>
<tr>
<td>7</td>
<td>Hopf bifurcation</td>
<td>44</td>
</tr>
<tr>
<td>7.1</td>
<td>Normalization</td>
<td>45</td>
</tr>
<tr>
<td>7.1.1</td>
<td>Linear terms</td>
<td>45</td>
</tr>
<tr>
<td>7.1.2</td>
<td>Maclaurin approximation</td>
<td>46</td>
</tr>
<tr>
<td>7.1.3</td>
<td>Removing of one nonlinear term</td>
<td>46</td>
</tr>
<tr>
<td>7.1.4</td>
<td>Removing of other nonlinear term</td>
<td>48</td>
</tr>
<tr>
<td>7.1.5</td>
<td>Back to real numbers</td>
<td>50</td>
</tr>
<tr>
<td>7.2</td>
<td>Study of the normal form</td>
<td>51</td>
</tr>
<tr>
<td>7.3</td>
<td>Theorem. Hopf bifurcation</td>
<td>51</td>
</tr>
<tr>
<td>7.4</td>
<td>Stability index $Re(c_1)$</td>
<td>52</td>
</tr>
<tr>
<td>8</td>
<td>Center manifold theory</td>
<td>55</td>
</tr>
<tr>
<td>8.1</td>
<td>Main theorems</td>
<td>55</td>
</tr>
<tr>
<td>8.2</td>
<td>Plan for computation of center manifold</td>
<td>56</td>
</tr>
<tr>
<td>8.3</td>
<td>System with a parameter</td>
<td>57</td>
</tr>
<tr>
<td>8.3.1</td>
<td>Center manifold for system with parameter</td>
<td>58</td>
</tr>
<tr>
<td>8.4</td>
<td>Fold bifurcation in a two-variable system</td>
<td>59</td>
</tr>
<tr>
<td>8.4.1</td>
<td>Theorem. Tangent (Saddle-node, Fold) bifurcation for two variable ODEs</td>
<td>60</td>
</tr>
<tr>
<td>8.4.2</td>
<td>Practical notes</td>
<td>61</td>
</tr>
<tr>
<td>8.5</td>
<td>Other bifurcations</td>
<td>61</td>
</tr>
<tr>
<td>9</td>
<td>1D maps. Main definitions</td>
<td>62</td>
</tr>
<tr>
<td>9.1</td>
<td>1D maps without parameter</td>
<td>62</td>
</tr>
<tr>
<td>9.2</td>
<td>1D map with a parameter</td>
<td>66</td>
</tr>
<tr>
<td>10</td>
<td>Fold bifurcation for maps</td>
<td>68</td>
</tr>
<tr>
<td>10.1</td>
<td>Normal form</td>
<td>68</td>
</tr>
<tr>
<td>10.1.1</td>
<td>New parameter</td>
<td>70</td>
</tr>
<tr>
<td>10.1.2</td>
<td>Rescaling of amplitude</td>
<td>70</td>
</tr>
<tr>
<td>10.1.3</td>
<td>Conclusion</td>
<td>71</td>
</tr>
<tr>
<td>10.2</td>
<td>Study of the normal form</td>
<td>71</td>
</tr>
<tr>
<td>10.3</td>
<td>Theorem. Tangent (Saddle-node, Fold) bifurcation for maps</td>
<td>72</td>
</tr>
<tr>
<td>11</td>
<td>Transcritical bifurcation</td>
<td>73</td>
</tr>
<tr>
<td>11.1</td>
<td>Normal form</td>
<td>73</td>
</tr>
<tr>
<td>11.2</td>
<td>Study of the normal form</td>
<td>74</td>
</tr>
<tr>
<td>11.3</td>
<td>Theorem. Transcritical bifurcation for maps</td>
<td>74</td>
</tr>
<tr>
<td>12</td>
<td>Pitchfork bifurcation for maps</td>
<td>76</td>
</tr>
<tr>
<td>12.1</td>
<td>Normal form</td>
<td>76</td>
</tr>
<tr>
<td>12.2</td>
<td>Study of the normal form</td>
<td>77</td>
</tr>
<tr>
<td>12.3</td>
<td>Theorem. Pitchfork bifurcation for maps</td>
<td>78</td>
</tr>
</tbody>
</table>
13 Flip bifurcation
13.1 Normal form ................................................. 81
13.1.1 Equilibrium shift ........................................ 81
13.1.2 Taylor expansion ....................................... 82
13.2 Study of the normal form .................................. 84
13.2.1 Study of the normal form of a single iterated map .... 84
13.3 Study of the normal form of a double iterated map ...... 85
13.3.1 Theorem. Flip (period doubling) bifurcation for maps. 86

14 Feigenbaum universality .................................. 88
14.1 Introduction .................................................. 88
14.1.1 The phenomenon ........................................ 88
14.2 Qualitative approach ...................................... 89
14.3 Quantitative approach .................................... 92

15 Maps in 2D ...................................................... 98
15.1 Linear maps .................................................. 98
15.1.1 Real eigen values ....................................... 98
15.1.2 Complex eigen values .................................. 98
15.2 Nonlinear maps .............................................. 99

16 Neimark-Sacker bifurcation ............................. 101
16.0.1 Linear terms. ............................................ 101
16.0.2 nonlinear terms ........................................ 101
16.1 Study of the normal form ................................. 103
16.1.1 Case \( d < 0 \) ........................................... 103
16.1.2 Case \( d > 0 \) ........................................... 104
16.2 Theorem. Neimark-Sacker bifurcation .................... 104

17 Center manifold for maps ................................. 105
17.1 Main theorems .............................................. 105
17.2 Plan for computation of center manifold ................. 106
17.3 Map with a parameter ................................... 107

18 Bifurcations of limit cycles of ODEs .................... 108

19 Maps on circle ............................................... 111
19.1 Shift map ................................................... 112
19.2 Non-linear map ............................................ 112
19.3 Rotation number ........................................... 113
19.4 Phase locking .............................................. 113
19.5 Phase locking and Neimark-Sacker bifurcation .......... 115

20 Homoclinic bifurcation in 2D ............................ 116

21 The cusp bifurcation ........................................ 121

22 Generalized Hopf bifurcation ............................ 125

23 Bogdanov-Takens bifurcation ............................ 132
### 24 PDE models in biology

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>24.1 Gradient</td>
<td>139</td>
</tr>
<tr>
<td>24.2 Main PDEs</td>
<td>140</td>
</tr>
<tr>
<td>24.2.1 conservation</td>
<td>140</td>
</tr>
<tr>
<td>24.2.2 convection</td>
<td>141</td>
</tr>
<tr>
<td>24.2.3 chemotaxis</td>
<td>142</td>
</tr>
<tr>
<td>24.2.4 diffusion</td>
<td>142</td>
</tr>
<tr>
<td>24.3 Numerical study of PDEs</td>
<td>142</td>
</tr>
<tr>
<td>24.3.1 approximations of derivatives</td>
<td>142</td>
</tr>
<tr>
<td>24.3.2 integration of an ODE</td>
<td>143</td>
</tr>
<tr>
<td>24.3.3 integration of a PDE</td>
<td>144</td>
</tr>
</tbody>
</table>

### 25 Turing instability

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>147</td>
</tr>
</tbody>
</table>
Chapter 1

ODE. Main definitions

Definition 1 Equation $\frac{dx}{dt} = f(x)$ is called the autonomous differential equation.

Definition 2 The problem $\frac{dx}{dt} = f(x), x(0) = x_0$ is called the initial value problem; Its solution is called the orbit or trajectory.

Definition 3 The collection of all orbits of a differential equation together with the direction arrows is called the phase portrait.

Definition 4 A point $x^*$ is called an equilibrium point of $\frac{dx}{dt} = f(x)$, if $f(x^*) = 0$.

Theorem 1 Suppose that $x^*$ is an equilibrium point of $\frac{dx}{dt} = f(x)$, then the equilibrium point $x^*$ is stable if $f'(x^*) < 0$, is non-stable if $f'(x^*) > 0$ and no information if $f'(x^*) = 0$.

Definition 5 The basin of attraction of a stable equilibrium point $x^*$ is the set of values of $x$ such that, if $x$ is initially somewhere in that set, it will subsequently move to the equilibrium point $x^*$.

Now, consider an ODE with a parameter $c$:

$$\frac{dx}{dt} = f(x, c)$$

An important question is, what happens with ODE if we change the parameter. There two possibilities: nothing changes, i.e., the qualitative behavior of the equation remains the same, or something changes, i.e., the ODE will have different qualitative behavior. To formulate it on mathematical language, we introduce the following definitions:

Definition 6 Two ODEs $\frac{dx}{dt} = f(x)$ and $\frac{dx}{dt} = g(x)$ are said to be topologically equivalent if there is a one-to one transformation $h$ such that $h$ takes the orbits of one ODE to the orbits of the other ODE and preserves the direction in time.

Definition 7 An equation $\frac{dx}{dt} = f(x, c)$ is called structurally stable at $c = c^*$ if $\frac{dx}{dt} = f(x, c)$ is topologically equivalent to $\frac{dx}{dt} = f(x, c^*)$ for all $c$ close to $c^*$. 
Now, we can formulate the plan of study of an ODE with a parameter:

**Plan of qualitative study**

1. Study ODE at some value of the parameter.
2. Apply a test of structural stability
3. If the system is structurally unstable study bifurcations.

The most important question here is what is the test of structural stability. In order to derive it, we need some extra mathematical background on theory of approximation of functions of one and two variables.
Chapter 2

Taylor series

2.1 One dimensional case

Taylor series give approximation of a function using derivatives of this function. The most simple type of approximation is a linear approximation which is closely connected to definition of the derivative. As we know the derivative of a function \( f(x) \) at a point \( x_0 \) is

\[
\frac{f(x) - f(x_0)}{x - x_0} = f'(x_0).
\]

If we write this formula without the limit, we get an approximate equality, instead of the exact equality:

\[
f'(x_0) \approx \frac{f(x) - f(x_0)}{x - x_0}.
\]

This formula will be more accurate if \( x \) will be closer to \( x_0 \). Further we find:

\[
f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0).
\]  

(2.1)

Let us check this formula by approximating the function \( y = 2x^2 + 1 \) at \( x_0 = 1 \). \( f(x_0) = 2 \cdot 1^2 + 1 = 3 \), \( f'(1) = 4 \), hence \( f(x) \approx 3 + 4 \cdot (x - 1) \). At \( x = 1.1 \) this approximate formula gives \( f(1.1) \approx 3 + 4 \cdot (1.1 - 1) = 3.4 \). The exact value is \( f(1.1) = 2 \cdot 1.1^2 + 1 = 3.42 \). So the error is just 0.6%. However, if \( x = 0 \), \( f(0) \approx 3 + 4 \cdot (0 - 1) = -1 \) while the exact value is \( f(0) = 1 \). So we see, that the approximate formula works good if \( x \) is close to \( x_0 \) only.

Equation (2.1) is the simplest form of Taylor series, which approximates the function using a polynomial of the degree one:

\[
a + bx = a + f'(x_0) \cdot (x - x_0).
\]

(2.2)

The idea of proof:

Let us derive the Maclaurin series up to the second order terms. We have a function \( f(x) \) and we want to approximate it by a quadratic polynomial: \( p(x) = a + bx + cx^2 \). Our idea is to find
such polynomial, which has the same value and the same 1st and 2nd derivatives as our function at the point \( x = 0 \). For that, let us consider \( a, b, c \) as unknown coefficient, and let us fix their values in order to satisfy our requirements.

1. The function must have the same value as the polynomial at \( x = 0 \):
\[
f(0) = p(0) = a + b \times 0 + c \times 0^2 = a
\]

therefore this condition determines the value of the first unknown coefficient \( a = f(0) \).

2. The function must have the same value of the first derivative as the polynomial at \( x = 0 \):
\[
f'(0) = p'(0) = b + 2c \times 0 = b
\]
Thus the value of the second unknown coefficient is \( b = f'(0) \).

3. The function must have the same value of the second derivative as the polynomial at \( x = 0 \):
\[
p''(x) = 2c, \quad f''(x) = f''(0) = 2c \quad \text{and this condition determines the value of the third unknown coefficient } c = \frac{f''(0)}{2}.
\]

Hence we found that the second order approximation of \( f(x) \) by a polynomial around the point \( x = 0 \) is:
\[
f(x) \approx a + bx + cx^2 = f(0) + f'(0) \times x + \frac{f''(0)}{2} \times x^2
\]

We see that this formula is exactly formula (2.3) up to the second second order. (Note, that \( 2! = 1 \times 2 = 2 \).

If we perform this procedure for a polynomial of the \( n \)-th degree, we will get formula (2.3).

### 2.2 Two dimensional case

#### 2.2.1 Linear approximation

Let us derive a similar formula for a function of two variables \( f(x, y) \). Let us assume, that we know \( f(x, y) \) and its partial derivatives at some point \( x_0, y_0 \) and we want to find the value of a function at the close point \( x, y \). Let us move to the point \( x, y \) in two steps. Let us first move

![Figure 2.1:](image)

from the point \( x_0, y_0 \) to the point \( x, y_0 \), i.e., in the \( x \)-direction, and then from \( x, y_0 \) to \( x, y \), i.e., in
the $y$-direction. We see, that on the first part of our motion $y$ is fixed at $y = y_0$, so the function is $f(x, y_0)$ and it depends just on one variable $x$. Let us denote $G(x) = f(x, y_0)$ and use the formula (2.1) to find the approximate value of the function $G(x)$:

$$G(x) = f(x, y_0) \approx G(x_0) + (\partial f / \partial x)(x - x_0), \quad (2.4)$$

as $G(x_0) = f(x_0, y_0)$ and $G'(x_0)$ equals the partial derivative of $f$ with respect to $x$, $\partial f / \partial x$, at the point $x_0, y_0$. So, we found an approximation for our function of two variables for the first part of our motion fig. 2.1. Now let us travel from $x, y_0$ to $x, y$. Here $x$ is fixed and $y$ changes from $y_0$ to $y$. We can easily make the same steps, and find the following formula similar to (2.4):

$$f(x, y) \approx f(x, y_0) + (\partial f / \partial y)(y - y_0), \quad (2.5)$$

where $\partial f / \partial y$ is the partial derivative of $f$ with respect to $y$ at the point $x, y_0$. Let us assume that $\partial f / \partial y$ at $(x, y_0)$ is approximately equal to $\partial f / \partial y$ at the initial point $(x_0, y_0)$. (The validity of such an approximation can be confirmed by a detailed analysis). The last step is to replace $f(x, y_0)$ in (2.5) by its approximation (2.4)). After that, we get the following approximation for the function of two variables:

$$f(x, y) \approx f(x_0, y_0) + (\partial f / \partial x)(x - x_0) + (\partial f / \partial y)(y - y_0) \quad (2.6)$$

This approximation is called linear, as the independent variables $x, y$ are in the first power only and we do not have terms like $x^2, y^2$, or $xy$.

### 2.2.2 Taylor series for a function of two variables

The following is a Taylor series for a function of two variables:

$$f(x, y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0)$$

$$+ \frac{\partial^2 f}{\partial x^2} \frac{(x - x_0)^2}{2} + \frac{\partial^2 f}{\partial x \partial y}(x - x_0)(y - y_0) + \frac{\partial^2 f}{\partial y^2} \frac{(y - y_0)^2}{2} + \cdots \quad (2.7)$$

$$+ \frac{1}{n!} \left( (x - x_0) \frac{\partial}{\partial x} + (y - y_0) \frac{\partial}{\partial y} \right)^n f$$

$$+ \cdots$$

The Maclaurin series, which gives the approximation of our function around the point 0, 0 is

$$f(x, y) \approx f(0, 0) + \frac{\partial f}{\partial x} x + \frac{\partial f}{\partial y} y$$

$$+ \frac{\partial^2 f}{\partial x^2} \frac{x^2}{2} + \frac{\partial^2 f}{\partial x \partial y} xy + \frac{\partial^2 f}{\partial y^2} \frac{y^2}{2} + \cdots \quad (2.8)$$

$$+ \frac{1}{n!} \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)^n f$$

$$+ \cdots$$

The main idea behind the Taylor series for a function of two variables is the same as for the Taylor series for a function of one variable: we try to approximate our function of two variables by polynomials of $x, y$ in such a way, that the approximation and our function have the same value and the same partial derivatives at the given point.
2.3 Change of variables

We will use Taylor series for qualitative study of differential equations. Our main method will be to expand the right hand sides of our equation close to equilibrium point into a Taylor series and study it. We have also seen, that Maclaurin series are simpler that Taylor series. It turns out, that we can almost always use Maclaurin series, instead of Taylor series in our computations.

We can understand it from the following simple consideration. Assume that we have a differential equation with an equilibrium point $x_0, c_0$, which is not located at the origin $0, 0$:

$$\frac{dx}{dt} = f(x, c) \quad f(x_0, c_0) = 0 \quad x_0 \neq 0 \quad c_0 \neq 0$$  \quad (2.9)

It turns out, that we can always shift this equilibrium point to the point $0, 0$ by change of variables. For that we just introduce a new variable $y(t) = x(t) - x_0$ and a new parameter $d = c - c_0$. Informally this means the following. Assume that in the original equation (2.9) we have the variable $x$ which accounts for the size of the population and the parameter $c$ which accounts for the birth rate of this population. Then an equilibrium $x_0, c_0$, can account, for example, for the ultimate size of the population at $c = c_0$. Introducing of a new variable $y(t) = x(t) - x_0$ mean that we want to measure not the actual size of the population, but the difference of the actual size from its ultimate value $x_0$. Similarly, a new parameter $d = c - c_0$ means that $d$ shows the difference in the birth rate compared to some value $c_0$. It is quite obvious, that the ultimate value of the new variable $y$ will be $y = 0$ at $d = 0$. Now, let us make a formal change of variables. The transformation of variables, or how we call it, the direct transformation is $y(t) = x(t) - x_0, d = c - c_0$. To make change of variables we use the inverse transformation: $x(t) = y(t) + x_0, c = d + c_0$, which shows us, how to find $x, c$ if we know the values of $y, d$. To make a transformation we replace $x, c$ by its values expressed in terms of new variables, and we get:

$$\frac{dx}{dt} = \frac{d(y + x_0)}{dt} = \frac{dy}{dt} = f(y + x_0, d + c_0) = F(y, d)$$

Using the direct transformation we will find, that equilibrium point $x_0, c_0$ will become:

$$y_0 = x_0 - x_0 = 0; \quad d_0 = c_0 - c_0 = 0$$

Hence we transformed our equation (2.9) to the new form with an equilibrium at the origin:

$$\frac{dy}{dt} = F(y, d) \quad F(0, 0) = 0$$  \quad (2.10)

Therefore, we can conclude, that for any differential equation we can assume that its equilibrium point which we want to study is located at the origin. This is because, if this equilibrium was initially not at the origin, we can always shift it there by the above transformation.

2.4 Implicit function theorem

Now let us use Taylor series to establish the first result on behavior of a system with a parameter.

**Theorem 2** Suppose that $\frac{dx}{dt} = f(x, c)$ has a stable (non-stable) equilibrium at $x = 0, c = 0$ with $\frac{\partial f}{\partial x}(0, 0) \neq 0$. Then there is a neighborhood of $c = 0$ in which our equation has an equilibrium point with the same stability.
Proof. The fact that $\frac{dx}{dt} = f(x,c)$ has an equilibrium at $x = 0, c = 0$ means $f(0, 0) = 0$. We have also assumed that $\frac{\partial f}{\partial x}(0, 0) \neq 0$.

Let us approximate our function around this equilibrium using the Taylor series. We know that the Taylor series have infinite number of terms, however, the main rule in such expansions, which we will widely use in this course, is the following: we keep the first nonzero terms, and drop the higher order terms. This is because we want to approximate our functions very close to the equilibrium point, and close to equilibrium the lower order terms are much larger than the higher order terms. To see it consider a Maclaurin approximation of the exponential function, for example. We know, that

$$e^x \approx 1 + x + \frac{x^2}{2} + \ldots$$

We claim that if we consider the values of $x$ which are closer and closer to $x = 0$ then the Maclaurin approximation of $e^x$ will be closer and closer to the lower power approximation $e^x \approx 1 + x$ and the extra term $\frac{x^2}{2}$ will have almost no effect on our approximation. We see it from the simple comparison, if $x = 0.1$ then the term of the first order in $x$ is $x = 0.1$, and the term of the second order in $x$ is $\frac{x^2}{2} = 0.005$. Thus the first order term is 20 times larger. However, if $x = 0.01$, $\frac{0.01^2}{2} = 0.00005$, and the first order term is 200 times larger, etc.

In general, the ratio of the first order term to the second order term is $\frac{x}{x^2} = \frac{1}{x}$, and we see that if $x$ goes to zero the first order term will dominate the second order term. Thus, if our aim is to study $e^x$ very close to $x = 0$, then the lower order term $x$ will have the leading effect, and we can neglect the second order term. Similarly, it is possible to show that the third, forth etc. order terms can be also neglected. Therefore the main conclusion is, that if we want to study the function $f(x)$ around point $x = 0$ using Maclaurin series, then we need to find the first non-vanishing term of the Maclaurin series and neglect all higher order terms.

In our case $f(0, 0) = 0$, but $\frac{\partial f}{\partial x}(0, 0) \neq 0$, therefore the first nonzero terms are of the first order, and we use Maclaurin expansion just up to the first order.

$$f(x, c) = f(0, 0) + \frac{\partial f}{\partial x} x + \frac{\partial f}{\partial c} c + \cdots$$

As we know $f(0, 0) = 0$. Because $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial c}$ are the values of partial derivatives at the point $(0, 0)$, they are just numbers, and we can denote them as: $\frac{\partial f}{\partial x} = a$ and $\frac{\partial f}{\partial c} = b$. Note, that $a \neq 0$, because we have assumed that $\frac{\partial f}{\partial x}(0, 0) \neq 0$.

Let us substitute our approximation into the differential equation. Note, that our approximation works good only close to the point around which we made our approximation, i.e., this approximation is valid only close to the point $x = 0, c = 0$. We get:

$$\frac{dx}{dt} = f(x, c) \approx f(0, 0) + \frac{\partial f}{\partial x} x + \frac{\partial f}{\partial c} c = ax + bc$$  \hspace{1cm} (2.11)

Now let us study system (2.11). To proof the theorem we need to show that equation (2.11) has an equilibrium point at $c$ close to $c = 0$, and that the stability of this equilibrium is the same as stability of the equilibrium at $c = 0$.

1. Find equilibria.

$$\begin{align*}
ax + bc &= 0 \\
x_{\text{equilibrium}} &= -\frac{b}{a}c
\end{align*}$$  \hspace{1cm} (2.12)

We see, that because $a \neq 0$ we can always find one equilibrium of our system close to $c = 0$. 

2. Stability of the equilibrium is given by the partial derivative of the right hand side of our equation with respect to $x$:

$$\frac{\partial(ax + bc)}{\partial x} = a$$

So we see that stability is the same as the origin.

So we proved the Theorem.

\[\text{Figure 2.2: Schematic bifurcation diagram for a hyperbolic stable (a) and non-stable (b) equilibrium}\]

We see, that the important property of equilibrium in our theorem was that $\frac{df}{dx}(x^*) \neq 0$. This property has a special name "hyperbolicity". The following is a formal definition of the hyperbolicity.

**Definition 8** An equilibrium point $x^*$ of $\frac{dx}{dt} = f(x)$ is called hyperbolic equilibrium if $\frac{df}{dx}(x^*) \neq 0$

Using this definition we can reformulate the theorem in the following way:

**Theorem 3** If $\frac{dx}{dt} = f(x,c)$ has a stable (non-stable) hyperbolic equilibrium at $x = 0, c = 0$, then there is neighborhood of $c = 0$ in which the ODE has an equilibrium with the same stability.

It is easy to see that this theorem gives us the test for structural stability of an ODE with respect to the change of the parameter in the “Plan of quantitative study” from chapter 1.

**Theorem 4** If $\frac{dx}{dt} = f(x,c)$ has a finite number of equilibria and they are hyperbolic then the ODE is structurally stable

**Bifurcation diagram**

Let us introduce one very important way of representation of results of studying of differential equations with parameter. This is called a bifurcation diagram.

The bifurcation diagram is a graph which shows the location of our equilibrium at different values of the parameter. For example, we can represent the results of the implicit function theorem using the following bifurcation diagram.

In fig.2.2 we presented two schematic diagrams. We use two axis, one for the parameter, and the second for a location of equilibrium. We represent the stable equilibria by solid lines and the non-stable equilibria by dashed lines. Because we do not know the stability of our equilibrium, (we just now that it is hyperbolic), we presented here the both possible cases. The lines show how the location of equilibrium changes with change of parameter value. In our case it is given by equation (2.12), i.e., we have a straight lines with some slope determined by $\frac{-b}{a}$. 
Chapter 3

Fold bifurcation

Using the “Plan of quantitative study” from chapter 1. we can study any ODE
\[
\frac{dx}{dt} = f(x, c)
\]  
(3.1)
around a hyperbolic point. This means, that if we check, that all equilibria of ODE (3.1) are hyperbolic, then nothing changes at small changes of the parameter \(c\). However, if we change the parameter further and further we can finally arrive to a parameter value \(c^*\) at which the ODE will have a non-hyperbolic equilibrium \((f(x^*, c^*) = 0; \frac{\partial f}{\partial x}(x^*, c^*) = 0\). What should we do in that case? The following chapter deals with this question.

3.1 Fold bifurcation on an example

Consider the following ODE model of a population growth:
\[
\frac{dy}{dt} = f(y, b) = 2 \frac{y^2}{1 + y^2} - b^2 y \quad b > 0
\]  
(3.2)
here the species cannot breed successfully when the numbers are too small or too large, and the death rate of spices is proportional to the square of concentration of a chemical \(b\).

In order to simplify further computations let us rewrite (3.2) in a slightly different form:
\[
\frac{dy}{dt} = f(y, b) = 2 - \frac{2}{1 + y^2} - b^2 y \quad b > 0
\]  
(3.3)
We can easily find that this equation has an equilibrium at \(y = 0\), which is always stable.

It turns out, that equation (3.3) can have other equilibria, and can even have a non-hyperbolic equilibrium. It occurs at \(b = 1, y = 1\).

To see it let us substitute \(b = 1, y = 1\) into the right hand side of equation (3.3) we find:
\[
f(1, 1) = 2 - \frac{2}{1 + 1^2} - 1^2 \times 1 = 2 - 1 - 1 = 0
\]
i.e., we do have an equilibrium at this point.

To check hyperbolicity of this equilibrium we compute the value of the partial derivative \(\frac{\partial f}{\partial y}(1, 1)\):
\[
\frac{\partial f}{\partial y} = \frac{4y}{(1 + y^2)^2} - b^2
\]
\[
\frac{\partial f}{\partial y}(1, 1) = \frac{4 + 1}{(1 + 1^2)^2} - 1^2 = 1 - 1 = 0
\]
We see that $\frac{\partial f}{\partial y}(1, 1) = 0$, hence the equilibrium is indeed a non-hyperbolic.

We know that we cannot study dynamics of our system around this equilibrium using methods from the previous chapter. Let us develop a new method of the analysis of ODE around a non-hyperbolic equilibrium point. The main idea of the analysis is to make several changes of variables and simplify our equation as much as we can. This simplest form of our ODE is called a normal form. Then we will study this normal form and make conclusions about the dynamics of our ODE.

Our first step in finding normal form for our ODE (3.3) will be the shift of equilibrium point $y = 1, b = 1$ to the point $0, 0$. The aim of this shift is the following. At the later stage we will use Taylor series to approximate the right hand side of our ODE around the equilibrium point. If the equilibrium will be at the point $0, 0$ we will be able to use the Maclaurin series, which are simpler than the Taylor series.

We proceed as we did in section 2.3. We introduce a new variable $x$ and a new parameter $c$ in the following way:

$$x = y - 1 \quad c = b - 1$$

We find that derivative of this new variable $x$ is:

$$\frac{dx}{dt} = \frac{d(y-1)}{dt} = \frac{dy}{dt}$$

On the other hand $\frac{dy}{dt}$ is given by (3.3):

$$\frac{dy}{dt} = 2 - \frac{2}{1+y^2} - b^2 y = 2 - \frac{2}{1+(x+1)^2} - (c + 1)^2(x + 1)$$

Thus ODE in new variables is:

$$\frac{dx}{dt} = f(x, c) = 2 - \frac{2}{1+(x+1)^2} - (c + 1)^2(x + 1)$$

(3.4)

It is easy to check that ODE (3.4) has a non-hyperbolic equilibrium at point $0, 0$, i.e., $f(0, 0) = 0 \quad \frac{\partial f}{\partial x}(0, 0) = 0$.

Our next step is to find a Maclaurin expansion of the right hand side of equation (3.4), i.e., an approximation of the right hand side around this non-hyperbolic equilibrium. Because the first derivative $\frac{\partial f}{\partial x}(0, 0) = 0$, we will expand our function up to the second order terms. The general expression is:

$$f(x, c) = f(0, 0) + \frac{\partial f}{\partial x}(0, 0) x + \frac{\partial f}{\partial c}(0, 0) c +$$

$$\frac{\partial^2 f}{\partial x^2} \frac{x^2}{2} + \frac{\partial^2 f}{\partial x \partial c} x c + \frac{\partial^2 f}{\partial c^2} \frac{c^2}{2} + O((x, c)^3)$$

(3.5)
In our particular case of ODE (3.4) we get:

\[ f(0, 0) = 2 - \frac{2}{1 + (0 + 1)^2} - (0 + 1)^2(0 + 1) = 2 - 1 - 1 = 0; \]

\[ \frac{\partial f}{\partial x} = -2(c + 1)(x + 1); \frac{\partial f}{\partial x}(0, 0) = -2(0 + 1)(0 + 1) = -2 \]

\[ \frac{\partial^2 f}{\partial x^2} = \frac{4}{(1 + (x + 1)^2)^2} - \frac{16(x + 1)^2}{(1 + (x + 1)^2)^3}; \frac{\partial^2 f}{\partial x^2}(0, 0) = \frac{4}{(2)^2} - \frac{16}{(2)^3} = -1 \]

Substituting these values to (3.5) we find:

\[ f(x, c) = -2c - \frac{x^2}{2} - 2xc - 2c^2 + O((x, c)^3) \]

Or the ODE becomes:

\[ \frac{dx}{dt} = -2c - \frac{x^2}{2} - 2xc - c^2 + O((x, c)^3) \]

It can be proved that the higher order terms which are collected in \( O((x, c)^3) \) are not important, so, we will omit them and our ODE around a non-hyperbolic equilibrium point becomes:

\[ \frac{dx}{dt} = -2c - \frac{x^2}{2} - 2xc - c^2 \]

How can we simplify this equation further? In many cases, there are no general rules for simplification, they were just found by trial and error method. However, in the case of equation (3.7) we can see the possible way of simplification from the following arguments. The right hand side of (3.7) is a quadratic polynomial with respect to \( x \), say \( Ax^2 + Bx + C \). The graph of this function is a parabola. We know, that by change of the \( x \) variable (i.e., by the change of the \( x \)-axis) we can shift parabola in a symmetric position with respect to the point \( x = 0 \), i.e., to transform it to the expression \( Ax^2 + C \). It is quite obvious that this new form is simpler than the original. It is also clear how to achieve it. We need to shift the \( x \)-axis by some value \( \delta \), i.e., introduce a new variable \( \xi \)

\[ \xi = x - \delta \]

At the moment we do not know the value of \( \delta \), however we will find it later in order to achieve our aim: to transform the right hand side of equation (3.7) to the form \( A\xi^2 + C \).

Transformation (3.8) is a direct transformation. We will also need the inverse transformation:

\[ x = \xi + \delta \]

Our plan will be the following:
1. Differentiate the direct transformation \( \xi(x, \delta) \). We will find an expression which will include \( x \) and its derivatives.

2. Express the derivatives of \( x \) using the differential equation (3.7).

3. Replace \( x \) in the right hand side by the inverse transformation \( x(\xi, \delta) \). After that we will get the ODE in the new coordinates.

4. Fix the unknown parameter \( \delta \) to simplify the equation.

Let us do it

1. Differentiate the direct transformation (3.8). Because \( \delta \) is a parameter (constant) we get:

\[
\frac{d\xi}{dt} = \frac{dx}{dt} - \frac{d\delta}{dt} = \frac{dx}{dt}
\]

2. Express the derivatives of \( x \) using the differential equation (3.7).

\[
\frac{d\xi}{dt} = \frac{dx}{dt} = -2c - \frac{x^2}{2} - 2xc - c^2
\]

3. Replace \( x \) in the right hand side by the inverse transformation (3.9)

As \( x = \xi + \delta \) we get

\[
\frac{d\xi}{dt} = -2c - \frac{(\xi + \delta)^2}{2} - 2(\xi + \delta)c - c^2
\]

\[
= (-2c - c^2 - 2\delta c - \frac{\delta^2}{2})
\]

\[
+\xi(-2c - \delta)
\]

\[
-\frac{1}{2}\xi^2
\]

4. Fix the unknown parameter \( \delta \) to simplify the equation.

Because we need to remove the linear terms \( \xi(-2c - \delta) \) we require:

\[
-2c - \delta = 0
\]
or
\[
\delta = -2c
\]  
(3.10)

After that our equation becomes:
\[
\frac{d\xi}{dt} = T_0 - \frac{1}{2} \xi^2
\]  
(3.11)

where
\[
T_0 = -2c - c^2 - 2\delta c - \frac{\delta^2}{2}
\]

If we substitute here the value of \(\delta = -2c\) from (3.10) and simplify the expression we get:
\[
T_0 = -2c + c^2
\]  
(3.12)

Now we will further simplify the equation. For that we will introduce a new parameter and make a transformation “rescaling of amplitude”

3.1.1 New parameter

Let us denote \(T_0\) as the new parameter \(\beta\). We can do it because at small values of \(c\) the changes of \(c\) and \(T_0\) are similar \((T_0 = -2c + c^2 \approx -2c, \text{at small } c)\). However, if we denote \(\beta = T_0 \approx -2c\), then \(\beta\) decreases with increase of \(c\). For that reason, let us also make a choice of the parameter \(\beta\) that change of \(\beta\) will follow change of the original parameter \(c\):
\[
\beta = -T_0 \approx 2c
\]  
(3.13)

Now our equation becomes:
\[
\frac{d\xi}{dt} = -\beta - \frac{1}{2} \xi^2
\]  
(3.14)

We see that equation became quite simple. However, we can make the last step in simplification and remove the factor \(\frac{1}{2}\) in equation (3.14).

3.1.2 Rescaling of amplitude

We introduce a new variable
\[
\eta = \frac{1}{2} \xi \quad \text{or} \quad \xi = 2\eta
\]  
(3.15)

our equation becomes:
\[
\frac{d\eta}{dt} = \frac{1}{2} \frac{d\xi}{dt} = -\beta - \frac{1}{4} \xi^2
\]  
(3.16)

After the inverse substitution \(\xi = 2\eta\) we get
\[
\frac{d\eta}{dt} = -\frac{1}{2} \beta - \frac{1}{4} \eta^2
\]  
(3.17)

If we denote \(\frac{1}{2} \beta\) as a new parameter \(\mu\) we get the following normal form
\[
\frac{d\eta}{dt} = -\mu - \eta^2
\]  
(3.18)

We see that differential equation (3.4) which has a non-hyperbolic equilibrium at \(y = 1, b = 1\) can be transformed to the more simple form (3.18).
3.1.3 Study of the normal form

Let study an ODE (3.18).

1. Equilibria.

\[ -\mu - \eta^2 = 0 \]

or

\[ \eta = \pm \sqrt{-\mu} \quad if \quad \mu < 0 \]

2. Stability

\[ \frac{\partial f}{\partial \eta} = -2\eta \]

\[ \frac{\partial f}{\partial \eta} (\sqrt{-\mu}) = -2\sqrt{-\mu} < 0 \]

i.e., equilibrium \( \eta = \sqrt{-\mu} \) is stable

\[ \frac{\partial f}{\partial \eta} (-\sqrt{-\mu}) = 2\sqrt{-\mu} > 0 \]

i.e., equilibrium \( \eta = -\sqrt{-\mu} \) is unstable

The bifurcation diagram if shown in fig.3.2.

![Bifurcation diagram](image)

Figure 3.2: Bifurcation diagrams for the fold bifurcation of equation 3.18

3.1.4 Conclusion

Now, let us make final conclusions about our ODE (3.3). As we mentioned above, it has always a stable equilibrium at \( x = 0 \). We denote it as a solid line. This solid line together with the bifurcation diagram from fig.3.2 will give us the final bifurcation diagram shown in fig.3.3. From the bifurcation diagram fig.3.3 we see that in our ODE we have two regions of different behavior: below the non-hyperbolic point at \( b = 1 \) we have three equilibria, two of which (the upper one and lower one) are stable, and the middle one is unstable. The dynamics of population in this region is, if its size is smaller than the position of a non-stable equilibrium it goes to extinction, if its size is more than that value, it goes to the upper equilibrium. However, if the concentration of \( b \) increases above the value \( b = 1 \) the upper two equilibria disappear and the only attractor in our ODE is \( x = 0 \), i.e., population will be extinct.
3.2 Normal form for the fold bifurcation

Now, let us consider the same approach for study an ODE around a non-hyperbolic equilibrium point for the general the ODE:

\[
\frac{dx}{dt} = f(x, c) \tag{3.19}
\]

Assume that \(f(x, c)\) has a non-hyperbolic equilibrium at \(x = 0, c = 0:\)

\[
f(0, 0) = 0 \quad \frac{df}{dx}(0, 0) = 0 \tag{3.20}
\]

Let us find a normal form for ODE (3.19) in a way similar to that of from the previous section. For that let us first find a Maclaurin expansion of the right hand side of equation (3.19)

\[
f(x, c) = f(0, 0) + \frac{\partial f}{\partial x}(0, 0)x + \frac{\partial f}{\partial c}(0, 0)c + \frac{\partial^2 f}{\partial x^2}x^2 + \frac{\partial^2 f}{\partial x\partial c}xc + \frac{\partial^2 f}{\partial c^2}c^2 + O((x, c)^3) \tag{3.21}
\]

Because of conditions (3.20) the expansion becomes:

\[
f(x, c) = \frac{\partial f}{\partial c}c + \frac{\partial^2 f}{\partial x^2}x^2 + \frac{\partial^2 f}{\partial x\partial c}xc + \frac{\partial^2 f}{\partial c^2}c^2 + O((x, c)^3) \tag{3.22}
\]

Or the ODE becomes:

\[
\frac{dx}{dt} = \frac{\partial f}{\partial c}c + \frac{\partial^2 f}{\partial x^2}x^2 + \frac{\partial^2 f}{\partial x\partial c}xc + \frac{\partial^2 f}{\partial c^2}c^2 + O((x, c)^3) \tag{3.23}
\]

We say as in the previous section that the higher order terms which are collected in \(O((x, c)^3)\) are not important for this bifurcation. So, we will omit them:

\[
\frac{dx}{dt} = \frac{\partial f}{\partial c}c + \frac{\partial^2 f}{\partial x^2}x^2 + \frac{\partial^2 f}{\partial x\partial c}xc + \frac{\partial^2 f}{\partial c^2}c^2 \tag{3.23}
\]

Now we make further normalization of the ODE. Our aim as in the previous section is to remove the term \(\frac{\partial^2 f}{\partial x\partial c}xc\) which is linear with respect to the variable \(x\).

We introduce a new variable \(\xi\)

\[
\xi = x - \delta \tag{3.24}
\]
where δ is unknown parameter which we will fix in order to make normalization. The inverse transformation is:

$$x = \xi + \delta$$  \hspace{1cm} (3.25)

Our plan of transformation is exactly the same as in the previous section:

1. Differentiate the direct transformation (3.24). Because δ is a parameter (constant) we get:

$$\frac{d\xi}{dt} = \frac{dx}{dt} - \frac{d\delta}{dt} = \frac{dx}{dt}$$

2. Express the derivatives of x using the differential equation (3.23).

$$\frac{d\xi}{dt} = \frac{dx}{dt} = \frac{\partial f}{\partial c}c + \frac{\partial^2 f}{\partial x^2} \frac{c^2}{2} + \frac{\partial^2 f}{\partial x \partial c} x c + \frac{\partial^2 f}{\partial c^2} $$

3. Replace x in the right hand side by the inverse transformation (3.25)

As $x = \xi + \delta$ we get

$$\frac{d\xi}{dt} = \frac{\partial f}{\partial c}c + \frac{\partial^2 f}{\partial x^2} \frac{c^2}{2} + \frac{\partial^2 f}{\partial x \partial c} \delta c + \frac{\partial^2 f}{\partial c^2}$$

$$+ \xi(\frac{\partial^2 f}{\partial x \partial c} + \frac{\partial^2 f}{\partial x^2})$$

$$+ \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \delta^2$$

4. Fix the unknown parameter δ to simplify the equation.

Because we need to remove the linear terms $+ \xi(\frac{\partial^2 f}{\partial x \partial c} + \frac{\partial^2 f}{\partial x^2})$ we require:

$$\frac{\partial^2 f}{\partial x \partial c} c + \frac{\partial^2 f}{\partial x^2} = 0$$

or

$$\delta = -\frac{\frac{\partial^2 f}{\partial x \partial c} c}{\frac{\partial^2 f}{\partial x^2}}$$  \hspace{1cm} (3.26)

We can do it if:

$$\frac{\partial^2 f}{\partial x^2} \neq 0$$  \hspace{1cm} (3.27)

After that our equation becomes:

$$\frac{d\xi}{dt} = T_0 + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \xi^2$$  \hspace{1cm} (3.28)

where

$$T_0 = \left( \frac{\partial f}{\partial c} c + \frac{\partial^2 f}{\partial x^2} \frac{c^2}{2} + \frac{\partial^2 f}{\partial x \partial c} \delta c + \frac{\partial^2 f}{\partial c^2} \right)$$

If we substitute here the value of δ from (3.26) and simplify the expression we get:

$$T_0 = \frac{\partial f}{\partial c} c + c^2 \left( \frac{\partial^2 f}{\partial c^2} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} - \frac{(\frac{\partial^2 f}{\partial x \partial c})^2}{2 \frac{\partial^2 f}{\partial x^2}} \right)$$  \hspace{1cm} (3.29)

As in the previous section we will introduce a new parameter and make a transformation “rescaling of amplitude”
3.2.1 New parameter

If
\[
\frac{\partial f}{\partial c} \neq 0
\]  
(3.30)

then we can neglect the terms in (3.29) which contain \(c^2\) and find

\[
T_0 \approx \frac{\partial f}{\partial c} c
\]

From here we see that \(T_0\) is proportional to \(c\). We also see that if \(\frac{\partial f}{\partial c} > 0\), then \(T_0\) increases with increase of \(c\), but if \(\frac{\partial f}{\partial c} < 0\), then \(T_0\) decreases with increase of \(c\). For that reason, let us also make a more definite choice of the parameter \(\beta\) that it will always follow the original parameter \(c\), i.e., it will always increase with increase of \(c\). We need it in order to keep the same direction of bifurcation in the old and in the new ODE. We can do it in the following way:

\[
\beta = T_0 \quad if \quad \frac{\partial f}{\partial c} > 0 \quad \beta = -T_0 \quad if \quad \frac{\partial f}{\partial c} < 0
\]  
(3.31)

Now our equation becomes:

\[
\frac{d\xi}{dt} = \pm \beta + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \xi^2
\]

(3.32)

3.2.2 Rescaling of amplitude

If we now introduce a new variable

\[
\eta = \left| \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right| \xi
\]

(3.33)

our equation becomes:

\[
\frac{d\eta}{dt} = \left| \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right| \frac{d\xi}{dt} = \pm \beta \left| \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right| + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \left| \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right| \xi^2
\]

(3.34)

After the inverse substitution \(\xi = \frac{\eta}{\left| \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right|}\) we get

\[
\frac{d\eta}{dt} = \pm \beta \left| \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right| + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \left| \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right| \eta^2
\]  
(3.35)

Note, that

\[
\frac{1}{2} \frac{\partial^2 f}{\partial x^2} = 1 \quad if \quad \frac{\partial^2 f}{\partial x^2} > 0 \quad and \quad \frac{1}{2} \frac{\partial^2 f}{\partial x^2} = -1 \quad if \quad \frac{\partial^2 f}{\partial x^2} < 0
\]  
(3.36)

If we denote \(\beta \left| \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right|\) as a new parameter \(\mu\) we get the following normal form

\[
\frac{d\eta}{dt} = \pm \mu \pm \eta^2
\]  
(3.37)

We see that any differential equation which has a non-hyperbolic equilibrium can be transformed to the more simple form (3.37). However, in order to make these transformation we need that conditions (3.27) (3.30) be satisfied. These conditions are called non-degeneracy conditions.
3.2.3 Study of the normal form

Let us consider the case \( \frac{d\eta}{dt} = \mu - \eta^2 \)

1. Equilibria.
\[
\mu - \eta^2 = 0
\]

or
\[
\eta = \pm \sqrt{\mu} \quad \text{if} \quad \mu > 0
\]

2. Stability
\[
\frac{\partial f}{\partial \eta} = -2\eta
\]
\[
\frac{\partial f}{\partial \eta}(\sqrt{\mu}) = -2\sqrt{\mu} < 0
\]
i.e., equilibrium \( \eta = \sqrt{\mu} \) is stable
\[
\frac{\partial f}{\partial \eta}(-\sqrt{\mu}) = 2\sqrt{\mu} > 0
\]
i.e., equilibrium \( \eta = -\sqrt{\mu} \) is unstable

3.3 Theorem. Tangent (Saddle-node, Fold) bifurcation for ODEs

Let \( \frac{dx}{dt} = f(x, c) \), has an equilibrium point \( x = x^*, c = c^* \), with
\[
\frac{\partial f}{\partial x}(x^*, c^*) = 0 \quad (3.38)
\]
and
\[
\frac{\partial f}{\partial c}(x^*, c^*) \neq 0 \quad \frac{\partial^2 f}{\partial x^2}(x^*, c^*) \neq 0 \quad (3.39)
\]
then close to \( (x^*, c^*) \) the equation is locally equivalent to one of the following normal forms:
\[
\frac{d\eta}{dt} = \pm \mu \pm \eta^2 \quad (3.40)
\]
and the tangent bifurcation takes place.

Note: the sign at \( \mu \) is the same as the sign of \( \frac{\partial f}{\partial c} \), and the sign at \( \eta^2 \) is the same as the sign of \( \frac{\partial^2 f}{\partial x^2} \).

Note, that the conditions of this bifurcation can be computed at the point of non-hyperbolicity \( x = x^*, c = c^* \) without shifting it to zero. This is because the linear shift does not affect the derivatives (3.38)-(3.40).
Bifurcation diagrams

\[ \eta' = \mu - \eta^2 \]

\[ \eta' = \mu + \eta^2 \]

\[ \eta' = -\mu - \eta^2 \]

\[ \eta' = -\mu + \eta^2 \]

Figure 3.4: Bifurcation diagrams for the fold bifurcation
Chapter 4

Transcritical bifurcation

In general systems non-degeneracy conditions for the fold bifurcation are usually satisfied. But there are several classes of ODEs in which these conditions are not valid. One such class is very often occurs in biological systems. Let us consider an ODE:

\[ \frac{dx}{dt} = f(x, c) = xg(x, c) \]

Such kind of equations often describes the growth of population. In such ODE \( x = 0 \) is always an equilibrium point (at \( x = 0 \) \( xg(x, c) = 0 \)). This fact has a simple biological meaning: if the population \( x \) is extinct it will be extinct forever.

We can study ODE around equilibrium \( x = 0 \) as usual: i.e., find hyperbolicity of this equilibrium:

\[ \frac{\partial f}{\partial x}(0, c) = \frac{\partial(xg)}{\partial x} = x \frac{\partial g}{\partial x}(0, c) + g(0, c) = g(0, c) \]

As we know, if \( \frac{\partial f}{\partial x}(0, c) = g(0, c) \neq 0 \) the equilibrium is hyperbolic and we do not expect any bifurcations around it.

However, if at some parameter value \( c^* \) the equilibrium becomes non-hyperbolic (\( g(0, c^*) = 0 \)) we expect a bifurcation. Note, that the fold bifurcation is not possible here. This is because at fold bifurcation we have a change from zero to two equilibria when we change the parameter value. However, because in our ODE \( x = 0 \) is always an equilibrium point, it exists before and after the bifurcation value of parameter \( c^* \), hence we cannot have fold bifurcation here. We can also see this from checking non-degeneracy conditions. As we know from the analysis of the fold bifurcation one of the non-degeneracy conditions is:

\[ \frac{\partial f}{\partial c}(x^*, c^*) \neq 0 \]

for our system we get:

\[ \frac{\partial f}{\partial c}(0, c^*) = \frac{\partial(xg)}{\partial c} = x \frac{\partial g}{\partial c}(0, c^*) = 0 \]

So this non-degeneracy condition is always not valid and we need to perform a special study of this case.

Let us assume that the value of the parameter when equilibrium \( x = 0 \) becomes non-hyperbolic is \( c^* = 0 \) and study what happens around this non-hyperbolic equilibrium of ODE.
4.1 Normal form

Consider the ODE:

\[
\frac{dx}{dt} = f(x, c) = xg(x, c) \tag{4.1}
\]

Assume that \(xg(x, c)\) has a non-hyperbolic equilibrium at \(x = 0, c = 0\):

\[
g(0, 0) = 0 \tag{4.2}
\]

Let us find a normal form for ODE (4.1). For that let us first find a Taylor expansion of the right hand side of equation (4.1). In this case we need the first order terms only:

\[
f(x, c) = xg(x, c) = x(g(0, 0) + \frac{\partial g}{\partial x}(0, 0)x + \frac{\partial g}{\partial c}(0, 0)c)
\]

\[
= x\left(\frac{\partial g}{\partial x}(0, 0)x + \frac{\partial g}{\partial c}(0, 0)c\right) \quad \text{as} \quad g(0, 0) = 0 \tag{4.3}
\]

It is obvious, that if we introduce a new parameter and make rescaling of the amplitude, in the same way as we did for the analysis of the fold bifurcation, we get the following normal form:

\[
\frac{d\eta}{dt} = \eta(\pm \mu \pm \eta) \tag{4.4}
\]

The obvious non-degeneracy conditions for normalization are:

\[
\frac{\partial g}{\partial x}(0, 0) \neq 0 \quad \frac{\partial g}{\partial c}(0, 0) \neq 0 \tag{4.5}
\]

4.2 Study of the normal form

Let us consider the case

\[
\frac{d\eta}{dt} = \eta(\mu - \eta)
\]

1. Equilibria.

\[
\eta(\mu - \eta) = 0
\]

or

\[
\eta = 0 \quad \eta = \mu
\]

2. Stability

\[
\frac{\partial f}{\partial \eta} = \mu - 2\eta
\]

\[
\frac{\partial f}{\partial \eta}(0) = \mu
\]

i.e., equilibrium \(\eta = 0\) is stable for \(\mu < 0\) and unstable for \(\mu > 0\).

The second equilibrium \(\eta = \mu\):

\[
\frac{\partial f}{\partial \eta}(\mu) = -\mu
\]

i.e., equilibrium \(\eta = \mu\) is stable for \(\mu > 0\) and unstable for \(\mu < 0\)
4.3 Theorem. Transcritical bifurcation for ODEs

Let $dx/dt = f(x,c) = xg(x,c)$, has a non-hyperbolic equilibrium point $x = 0, c = 0$:

$$g(0,0) = 0$$

and

$$\frac{\partial g}{\partial c}(0,0) \neq 0 \quad \frac{\partial g}{\partial x}(0,0) \neq 0$$

then close to $(0,0)$ the equation is locally equivalent to one of the following normal forms:

$$d\eta/dt = \pm \mu \eta \pm \eta^2$$

and the transcritical bifurcation takes place.

Note: the sign at $\mu$ is the same as the sign of $\frac{\partial g}{\partial c}$, and the sign at $\eta^2$ is the same as the sign of $\frac{\partial g}{\partial x}$.

Bifurcation Diagrams

![Bifurcation Diagrams](image-url)

Figure 4.1: Bifurcation diagrams for the transcritical bifurcation
Chapter 5

Pitchfork bifurcation

5.1 Odd functions and their derivatives

In this chapter we consider another bifurcation which occurs around non-hyperbolic equilibrium point for so-called odd functions.

**Definition 9** A function $f(x)$ is called an odd function if $f(-x) = -f(x)$.

There are many examples of odd functions, e.g. $\sin(x)$, of $x$ in any odd power $x^3, x^{2k+1}$. There is another important class of function which are called even.

**Definition 10** A function $f(x)$ is called an even function if $f(-x) = f(x)$.

Examples of even functions are $\cos(x), x$ in any even power $x^2, x^{2k}$, etc.

The following two properties of odd functions are important for the theory bifurcations.

1. $f(0) = 0$ if $f(x)$ is odd function

or in other words $x = 0$ is always an equilibrium point of the ODE $\frac{dx}{dt} = f(x)$.

To see it, we just apply the main property of the odd function $f(-x) = -f(x)$ for the point $x = 0$:

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = 0
\]

or in other words $x = 0$ is always an equilibrium point of the ODE $\frac{dx}{dt} = f(x)$.

2. The derivative of odd function is even function; The derivative of even function is odd function.

To see it, compute the first derivative of the odd function $f(x)$ at two symmetric point: $x$ and at $(-x)$

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

and

\[
f'(-x) = \lim_{h \to 0} \frac{f(-x) - f(-x - h)}{h}
\]

Now, let us rewrite an expression for under the sign of the limit for $f'(-x)$, using the fact, that $f'(x)$ is an odd function:
\[
\frac{f(-x) - f(-x-h)}{h} = \frac{-f(x) + f(x+h)}{h} = \frac{f(x+h) - f(x)}{h}
\]

Therefore, from the above definitions (5.2,5.3) we conclude that \(f'(-x) = f'(x)\), or that the derivative of the odd function is an even function.

In order to complete the proof we need to show now, that the derivative of an even function is an odd function. We proceed in the similar way. We assume that \(g(x)\) is an even function \(g(-x) = g(x)\), and we find its derivative in a similar way:

\[
g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \tag{5.4}
\]

\[
g'(-x) = \lim_{h \to 0} \frac{g(-x) - g(-x-h)}{h} \tag{5.5}
\]

Now, let us rewrite an expression for under the sign of the limit for \(g'(-x)\), using the fact, that \(g(x)\) is an even function:

\[
\frac{g(-x) - g(-x-h)}{h} = \frac{g(x) - g(x+h)}{h} = \frac{-g(x+h) - g(x)}{h}
\]

Therefore, from the above definitions of (5.4,5.5) we conclude that \(g'(-x) = -g'(x)\), or that the derivative of the even function is an odd function.

Now we can make a key observation regarding Maclaurin series of the odd functions. To find the series for \(f(x)\) we need to compute the derivatives of the function \(f(x)\) at \(x = 0\). If our function is odd, then we know from the above property (5.1) that \(f(0) = 0\). The first derivative \(f'(x)\) will be an even function of \(x\) and we have nothing special at \(x = 0\). However, the derivative of this even function \(f'(x)\) (i.e., \(f''(x)\)) is an odd function This means that \(f''(0) = 0\). It is easy to extent such analysis for the next derivatives. We will get the main result, that all even derivatives of the odd function at \(x = 0\) are zeros:

\[
f^{2k}(0) = 0 \quad k = 0, 1, 2, \ldots \tag{5.6}
\]

Finally if we have an odd function of two variables \(f(-x,c) = -f(x,c)\), then all the results will be the same. What we need to do, is just to replace the usual derivatives with partial derivatives:

\[
\frac{\partial^{2k} f}{\partial x^{2k}} (0) = 0 \quad k = 0, 1, 2, \ldots \tag{5.7}
\]

Note, also, that any partial derivative of \(f(x,c)\) with respect to \(c\) will not change the symmetry, i.e., if \(f\) is an odd function with respect to \(x\), then \(\frac{\partial f}{\partial c}\) will be also odd, and if \(f\) is an even function with respect to \(x\), then \(\frac{\partial f}{\partial c}\) will be even. You can easily see it by constructing expressions for the derivatives similar to (5.2,5.2). Thus, in some way, nothing depends on derivatives with respect to \(c\), and we can just formally add any number of such derivatives to the equations (5.7) without any further changes:

\[
\frac{\partial^{M+2k} f}{\partial x^{2k} \partial c^M} (0) = 0 \quad M, k = 0, 1, 2, \ldots \tag{5.8}
\]

Now, let us consider bifurcations of ODEs with odd functions at the right hand side.
5.2 Bifurcations of ODEs with odd functions

Now assume that we study bifurcations for an ODE with an odd function at the right hand side

\[ \frac{dx}{dt} = f(x, c) \quad f(-x, c) = -f(x, c) \quad (5.9) \]

Because of the property (5.1) \( f(0, c) = 0 \), and \( x = 0 \) will be an equilibrium point of this ODE for all values of the parameter \( c \).

Therefore we can immediately conclude that we cannot have a fold bifurcation for the equilibrium \( x = 0 \) of equation (5.9). This is because the fold bifurcation means that at some value of the parameter a pair of equilibrium appears/disappears, while in equation (5.9) the equilibrium \( x = 0 \) persists for all values of \( c \).

We can also see it formally, from the theorem on the fold bifurcation. In order to have a fold bifurcation at \( x = 0 \) and \( c = c^* \) we need to satisfy the necessary condition:

\[ \frac{\partial f}{\partial c}(0, c^*) = 0 \]

and two non-degeneracy conditions:

\[ \frac{\partial^2 f}{\partial x^2}(0, c^*) \neq 0 \quad \frac{\partial^2 f}{\partial x^2}(0, c^*) \neq 0 \quad (5.10) \]

The necessary condition can be satisfied at some parameter value. However, as we can see in a moment the both non-degeneracy conditions do not hold: \( \frac{\partial f}{\partial c}(0, c^*) \) involves zero derivative of the odd function with respect to \( x \), therefore from the equation (5.8) with \( k = 0, M = 1 \) we conclude, that \( \frac{\partial f}{\partial c}(0, c^*) = 0 \). Similarly \( \frac{\partial f}{\partial c^2}(0, c^*) = 0 \), as it involves the second derivative with respect to \( x \) at \( x = 0 \). Thus, the both non-degeneracy conditions are not satisfied, and we cannot have a fold bifurcation here. Therefore we need to make again a normalization of the right hand side of our equation around a non-hyperbolic point.

5.3 Normal form

Consider the ODE:

\[ \frac{dx}{dt} = f(x, c) \quad (5.11) \]

Assume that \( f(x, c) \) is an odd function:

\[ f(-x, c) = -f(x, c) \quad (5.12) \]

and that ODE (5.11) has a non-hyperbolic equilibrium at \( x = 0, c = 0 \)

\[ f(0, c) = 0, \quad \frac{\partial f}{\partial x}(0, 0) = 0. \quad (5.13) \]

Let us find a normal form for ODE (5.11). Note, that because \( f \) is an odd function, all terms in Taylor series which involve \( x \) to the even powers (i.e., 0, 2, 4, .. etc.) equal to zero.

Taylor expansion of the right hand side is:

\[
f(x, c) = f(0, 0) + \frac{\partial f}{\partial x}(0, 0)x + \frac{\partial f}{\partial c}(0, 0)c + \\
\frac{\partial^2 f}{\partial x^2} \frac{x^2}{2} + \frac{\partial^2 f}{\partial x \partial c} xc + \frac{\partial^2 f}{\partial c^2} \frac{x^2}{2} + \\
\frac{\partial^3 f}{\partial x^3} \frac{x^3}{6} + \frac{\partial^3 f}{\partial x^2 \partial c} \frac{x^3}{2} + \frac{\partial^3 f}{\partial x \partial c^2} \frac{c^2}{2} + \frac{\partial^3 f}{\partial c^3} \frac{c^3}{6} \ldots
\]

(5.14)
Because our function is odd the derivatives as even powers of \(x\) are zeros (5.8):

\[
f(0,0) = 0 \quad \frac{\partial f}{\partial c} = 0 \quad \frac{\partial^2 f}{\partial x^2} = 0 \quad \frac{\partial^2 f}{\partial x \partial c} = 0 \quad \frac{\partial^3 f}{\partial x^3} = 0 \quad \frac{\partial^3 f}{\partial x^2 \partial c} = 0
\] (5.15)

Because equilibrium is non-hyperbolic \(\frac{\partial f}{\partial t} = 0\).

Substituting these into (5.14) yields:

\[
f(x,c) = x\left(\frac{\partial^2 f}{\partial x \partial c} c + \frac{\partial^3 f}{\partial x^2 \partial c} \frac{c^2}{2}\right) + \frac{\partial^3 f}{\partial x^3} \frac{x^3}{6}
\] (5.16)

Now, let us introduce a new parameter

\[
\beta = \frac{\partial^2 f}{\partial x \partial c} c + \frac{\partial^3 f}{\partial x^2 \partial c} \frac{c^2}{2} \quad \text{for} \quad \frac{\partial^2 f}{\partial x \partial c} > 0
\]

or

\[
-\beta = \frac{\partial^2 f}{\partial x \partial c} c + \frac{\partial^3 f}{\partial x^2 \partial c} \frac{c^2}{2} \quad \text{for} \quad \frac{\partial^2 f}{\partial x \partial c} < 0
\]

we get

\[
\frac{dx}{dt} = \pm \beta x + \frac{\partial^3 f}{\partial x^3} \frac{x^3}{6}
\] (5.17)

and after rescaling the amplitude we get the following normal form:

\[
\frac{d\eta}{dt} = \pm \mu \eta \pm \eta^3
\] (5.18)

The obvious non-degeneracy conditions for this case are:

\[
\frac{\partial^2 f}{\partial x \partial c}(0,0) \neq 0 \quad \frac{\partial^3 f}{\partial x^3}(0,0) \neq 0
\] (5.19)

### 5.4 Study of the normal form

Let us consider the case

\[
\frac{d\eta}{dt} = \mu \eta - \eta^3
\]

1. Equilibria.

\[
\mu \eta - \eta^3 = 0
\]

or

\[
\eta = 0; \quad \eta = \pm \sqrt{\mu} \quad \text{if} \quad \mu > 0
\]

2. Stability

\[
\frac{\partial f}{\partial \eta} = \mu - 3\eta^2
\]

\[
\frac{\partial f}{\partial \eta}(0) = \mu
\]

i.e., equilibrium \(\eta = 0\) is stable at \(\mu < 0\) and is unstable at \(\mu > 0\)

\[
\frac{\partial f}{\partial \eta} = \mu - 3\eta^2
\]

\[
\frac{\partial f}{\partial \eta}(\pm \sqrt{\mu}) = \mu - 3\mu = -2\mu < 0 \quad \text{for} \quad \mu > 0
\]

i.e., both equilibria \(\eta = \pm \sqrt{\mu}\) are stable.
3. Note, that at $\mu = 0$ the equilibrium is stable. In general the stability of the equilibrium is determined by the sign at $\pm \eta^3$, i.e., if the sign is negative the equilibrium at $\mu = 0$ is stable, if the sign is positive the equilibrium at $\mu = 0$ is unstable. The sign at $\pm \eta^3$ is determined by the sign of $\frac{\partial^3 f}{\partial x^3}$ (see (5.17)).

5.5 Theorem. Pitchfork bifurcation for ODEs.

Consider the equation $dx/dt = f(x, c)$, such that $f(-x, c) = -f(x, c)$ for all $c$ close to $c = 0$. If this equation has a non-hyperbolic equilibrium point at $x = 0, c = 0$:

$$\frac{\partial f}{\partial x}(0,0) = 0$$

and if

$$\frac{\partial^2 f}{\partial x \partial c}(0,0) \neq 0 \quad \frac{\partial^3 f}{\partial x^3}(0,0) \neq 0$$

then close to $(0,0)$ this equation is locally equivalent to one of the following normal forms:

$$d\eta/dt = \eta(\pm \mu \pm \eta^2)$$

and the pitchfork bifurcation takes place.

1. Note: the sign at $\mu$ is the same as the sign of $\frac{\partial^2 f}{\partial x \partial c}$, and the sign at $\eta^2$ is the same as the sign of $\frac{\partial^3 f}{\partial x^3}$.

2. Note, that stability of equilibrium at bifurcation point determines the stability of the new born solution, i.e., if the equilibrium at $\mu = 0$ is stable, than the 2 new equilibria are also stable and vise versa.

3. Note, that the cases, when the equilibria at $x \neq 0$ are stable, are called supercritical pitchfork bifurcation, the cases when the equilibria at $x \neq 0$ are unstable, are called sub-critical pitchfork bifurcation.
Figure 5.1: Bifurcation diagrams for the pitch-fork bifurcation
Chapter 6

Differential equations of two variables

This chapter contains some extract of information on differential equation of two variables based on the reader “Introduction to differential equations of one and two variables”. Please look there for more detailed information.

Consider a general system of two differential equations:

\[
\begin{align*}
\frac{dx}{dt} &= f(x,y,t) \\
\frac{dy}{dt} &= g(x,y,t)
\end{align*}
\]  

(6.1)

If the right hand side of (6.1) does not depend on time explicitly, the system is called autonomous:

\[
\begin{align*}
\frac{dx}{dt} &= f(x,y) \\
\frac{dy}{dt} &= g(x,y)
\end{align*}
\]  

(6.2)

Initial conditions specify the values of \(x\) and \(y\) at some moment of time (usually at \(t_0 = 0\)):

\[
x(0) = x_0 \quad y(0) = y_0
\]  

(6.3)

The system (6.1) with initial conditions (6.3) is called the initial value problem. It usually has a unique solution.

The general solution of (6.2) depends on two arbitrary constants.

Definition 11 A point \((x^*,y^*)\) is called an equilibrium point of a system (6.5) if

\[
f(x^*,y^*) = 0, \quad g(x^*,y^*) = 0
\]  

(6.4)

6.1 Linearization

\[
\begin{align*}
\frac{dx}{dt} &= f(x,y) \\
\frac{dy}{dt} &= g(x,y)
\end{align*}
\]  

(6.5)

Assume that the system has an equilibrium point at \((x^*,y^*)\).

\[
\begin{align*}
f(x^*,y^*) &= 0 \\
g(x^*,y^*) &= 0
\end{align*}
\]  

(6.6)

the Jacobian is

\[
J = \begin{pmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{pmatrix}
\]  

(6.7)
As it was shown in the reader “Introduction to differential . . .”, Chapter 4, we can approximate
system (6.5) around equilibrium (6.6) by the following linearized system:

\[
\begin{pmatrix}
\dot{u} \\
\dot{v}
\end{pmatrix}
= 
\begin{pmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
= J
\begin{pmatrix}
u \\
v
\end{pmatrix}
\tag{6.8}
\]

This linear system (6.8) has an equilibrium point at \(u = 0, v = 0\). The general anticipation
is that the phase portrait of the linear system around its equilibrium point \(u = 0, v = 0\) is close
to the phase portrait of the original system (6.5) around equilibrium (6.6). However, the exact
statement will be given in the following chapter.

The good thing about the linear system is that we can always solve it analytically.

6.2 General solution of linear system

Consider a general linear system of ODEs:

\[
\begin{cases}
\frac{dx}{dt} = ax + by \\
\frac{dy}{dt} = cx + dy
\end{cases}
\tag{6.9}
\]

As it was shown in the reader “Introduction to differential . . .”, the general solution of a
linear system (6.9) is given by:

\[
\begin{pmatrix}
x \\
y
\end{pmatrix}
= C_1 \begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} e^{\lambda_1 t} + C_2 \begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix} e^{\lambda_2 t}
\tag{6.10}
\]

where \(\lambda_1, \lambda_2\) are eigen values of the matrix

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]

and \(\begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix}, \begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix}\) are the corresponding eigen vectors.

6.2.1 Canonical, or Jordan normal form

If \(\lambda_1 \neq \lambda_2\) are the eigen values of the matrix \(A\) and \(v_1\) corresponds to \(\lambda_1\) and \(v_2\) corresponds to
\(\lambda_2\), then the matrix which columns are the vectors \(v_1\) and \(v_2\) transforms the matrix \(A\) into the
canonical form \(D\) by the transformation:

\[
D = T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}
\tag{6.11}
\]

In the canonical coordinate system given by transformation \(T\) the differential equations
(6.9) will be:

\[
\begin{align*}
\frac{du}{dt} &= \lambda_1 u \\
\frac{dv}{dt} &= \lambda_2 v
\end{align*}
\]

For linear system we can easily construct phase portraits around the equilibrium point on
the basis of the general solution (6.10). The details can be found in the reader “Introduction to
differential . . .”. The following section gives a short description of the phase portraits for the
real eigen values.
6.3 Equilibria types for real eigen values

Conclusion 1 Figure 6.1 shows the phase portrait of a saddle point. Such types of phase portrait occur close to equilibrium, at which eigen values and eigen vectors of the system are real and have different signs, i.e., $\lambda_1 < 0; \lambda_2 > 0$, or $\lambda_1 > 0; \lambda_2 < 0$. The phase portrait of a saddle point has two manifolds directed along the eigen vectors of the linearized system. One manifold is stable (corresponding to the negative eigen value of the system). The other manifold is non-stable (corresponding to the positive eigen value of the system).

Conclusion 2 If the eigen values of system (6.9) are real and both positive $\lambda_1 > 0, \lambda_2 > 0$ we have an equilibrium point called a non-stable node.

To draw a phase portrait at this equilibrium we need to show two non-stable manifolds along the eigen vectors of system (6.9) and add several diverging trajectories between the manifolds.
Conclusion 3 If the eigen values of system (6.9) are real and both negative \( \lambda_1 < 0, \lambda_2 < 0 \) we have an equilibrium point called a stable node.

To draw a phase portrait at this equilibrium we need to show two stable manifolds along the eigen vectors of system (6.9) and add several trajectories converging to the equilibrium (0,0).

![Phase portrait of a stable node](image)

Figure 6.3: A non-stable node

Finally, in order to draw correctly the phase portraits of the nodes, we need to know to which direction the most of trajectories are tangent. The following simple rule can be found from the equation (6.10)

Conclusion 4 For an equilibrium point of the type “node” the most of the trajectories are tangent to the eigenvector corresponding to the eigenvalue closest to 0.

This means that if, for example the eigen values are \( \lambda_1 = -0.5, \lambda_2 = -2 \), then the trajectories will be tangent to the eigenvector corresponding the eigen value \( \lambda_1 \); if, for example the eigen values are \( \lambda_1 = 1.5, \lambda_2 = 0.3 \), then the trajectories will be tangent to the eigenvector corresponding the eigen value \( \lambda_2 \);

6.4 General solution of a system with complex eigenvalues

It turns out that if \( \lambda_{12} = \alpha \pm i\beta \) the formula (6.10) is still valid, and the solution can be represented in the form:

\[
\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} e^{\lambda_1 t} + C_2 \begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix} e^{\lambda_2 t} \tag{6.12}
\]

but as \( \lambda_1 = \alpha + i\beta; \lambda_2 = \alpha - i\beta \) we get

\[
\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} e^{(\alpha + i\beta)t} + C_2 \begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix} e^{(\alpha - i\beta)t} \tag{6.13}
\]

This expression gives the correct solutions of (6.9). However, the form of the solution is not good. First, because \( \lambda_{12} \) are complex, then the eigenvectors \( \begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} \) and \( \begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix} \) will also be complex. Next, we should consider \( C_1, C_2 \) as general complex constants. Therefore (6.13) is a quite complicated expression which gives \( x \) and \( y \) as complex functions of time \( t \). Mathematically it is correct and if we substitute (6.13) into the original equation (6.9) we get an equality. However, we need to draw a phase portrait of (6.9) in the \( Oxy \)-plane, i.e., we need just real solutions. So, we need to extract a very small part of the general expression (6.13). This turns out to be not a simple task. And the first question here is: what is the complex function \( e^{i\beta t} \)?
**Euler formula**

The Euler formula gives a representation of $e^{\beta t}$ in terms of trigonometric functions. It is quite unexpected:

$$e^{\beta t} = \cos \beta t + i \sin \beta t$$  \hspace{1cm} (6.14)

or in another representation:

$$e^{i\phi} = \cos \phi + i \sin \phi$$  \hspace{1cm} (6.15)

When you see this formula for the first time it looks quite crazy. We know that $\sin \phi$ and $\cos \phi$ come from the simple geometry of triangles, $i = \sqrt{-1}$ and $e$ is a special exponential function. Why are these functions connected together in such a simple way (6.15)?

To prove this formula, one should use Taylor series. Here I will present another method for the justification of (6.15) on the basis of differential equations.

In the reader “Introduction to differential …”, in order to find a solution of a two-dimensional system, we first considered a one dimensional differential equation $\frac{dx}{dt} = ax$, and we found its solution $Ce^{at}$. Consider the following initial value problem for this equation:

$$\frac{dx}{dt} = ax \quad x(0) = 1$$  \hspace{1cm} (6.16)

This initial value problem has the unique solution $x(t) = e^{at}$. So we say, that the solution of (6.16) is $e^{at}$. But we can also say it vice versa: we can define the exponential function $e^{at}$ as the function which satisfies the initial value problem (6.16). And this is correct. For example, if we give to a person just this equation and a computer, he will be able to solve it and to draw the graph of $e^{at}$, even without knowledge about exponential functions. The advantage of such a definition is that it can be easily extended to complex numbers. So, let us define $e^{it}$ as a function which satisfies the initial value problem (6.16) with $a = i$

$$\frac{dx}{dt} = ix \quad x(0) = 1$$  \hspace{1cm} (6.17)

In other words: $e^{it}$ must be the function $x(t)$, such that $x(0) = 1$, and the derivative of this function $\frac{dx}{dt}(t)$ must be equal to this function times $i$, i.e., $\frac{dx}{dt}(t) = ix(t)$. Let us find an expression which satisfies these conditions. It turns out that it will be $x(t) = \cos t + i \sin t$. Let us check it. The first condition is satisfied:

$$x(0) = \cos 0 + i \sin 0 = 1 + 0i = 1,$$

To check the second condition we write:

$$\frac{dx}{dt}(t) = (\cos t + i \sin t)' = \cos' t + i \sin' t =$$

$$-\sin t + i \cos t$$

if we replace $-1$ by $i^2$ we get:

$$\frac{dx}{dt}(t) = -\sin t + i \cos t = i^2 \sin t + i \cos t =$$

$$i(\cos t + i \sin t) = ix(t),$$
i.e., the second condition is also satisfied. So the function \( x(t) = \cos t + i\sin t \) gives the solution of (6.17), hence it is the same as \( e^{at} \) or \( e^{it} = \cos t + i\sin t \) and we get the Euler formula (6.15). The formula (6.14) is just the formula (6.15) in which instead of \( \phi \) the letters \( \beta t \) are used. To find \( e^{(\alpha+i\beta)t} \) we write:

\[
e^{(\alpha+i\beta)t} = e^{\alpha t} e^{i\beta t} = e^{\alpha t} (\cos \beta t + i\sin \beta t)
\]

(6.18)

**General solution**

Now let us find a solution of a system with imaginary eigenvalues. As we know the general solution is given by the formula (6.13) and because of the Euler formula we can rewrite it in the following way:

\[
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
= C_1 \begin{pmatrix}
  v_{1x} \\
  v_{1y}
\end{pmatrix}
 e^{\alpha t} (\cos \beta t + i\sin \beta t) + C_2 \begin{pmatrix}
  v_{2x} \\
  v_{2y}
\end{pmatrix}
 e^{\alpha t} (\cos \beta t - i\sin \beta t),
\]

(6.19)

where \( C_1, C_2 \) are arbitrary complex constants and \( \begin{pmatrix}
  v_{1x} \\
  v_{1y}
\end{pmatrix} \) and \( \begin{pmatrix}
  v_{2x} \\
  v_{2y}
\end{pmatrix} \) are complex eigen vectors. Now we should find a real part of this complicated expression and get a general real solution of our system in this case. We know that a general solution of a system of two differential equations should depend on two arbitrary constants. We will use this fact to find the general solution. Our idea is instead of extracting all real solutions from (6.19) we will find just two real solutions, which by multiplying by two arbitrary constants will give the general solution.

We will be able to find these two solutions from the first term of (6.19):

\[
Y_1 = \begin{pmatrix}
  v_{1x} \\
  v_{1y}
\end{pmatrix} e^{\alpha t} (\cos \beta t + i\sin \beta t) = v_1 e^{\alpha t} (\cos \beta t + i\sin \beta t)
\]

(6.20)

Let us extract real and an imaginary parts of this term. If we use the formula for the eigen value \( \lambda = \alpha + i\beta \) we find the eigen vector \( v_1 \): It has the real and imaginary parts:

\[
v_1 = \begin{pmatrix}
  v_{1x} \\
  v_{1y}
\end{pmatrix} = \begin{pmatrix}
  -b \\
  a - \alpha - i\beta
\end{pmatrix} = \begin{pmatrix}
  -b \\
  a - \alpha
\end{pmatrix} + i \begin{pmatrix}
  0 \\
  -\beta
\end{pmatrix} = v_r + iv_i
\]

(6.21)

The vector \( v_1 \) has the real part \( v_r \) and imaginary part \( v_i \). So, the term (6.20) can be written as:

\[
Y_1 = (v_r + iv_i) e^{\alpha t} (\cos \beta t + i\sin \beta t) =
\]

\[
e^{\alpha t} (v_r \cos \beta t + iv_r \sin \beta t + iv_i \cos \beta t - v_i \sin \beta t)
\]

\[
= e^{\alpha t} (v_r \cos \beta t - v_i \sin \beta t) + ie^{\alpha t} (v_r \sin \beta t + v_i \cos \beta t)
\]

If we denote:

\[
y_1 = e^{\alpha t} (v_r \cos \beta t - v_i \sin \beta t)
y_2 = e^{\alpha t} (v_r \sin \beta t + v_i \cos \beta t)
\]

(6.22)

the term (6.20) can be rewritten as

\[
Y_1 = y_1 + iy_2.
\]

Let us prove that both \( y_1 \) and \( y_2 \) are the real solutions of (6.9). For that we will use the fact that (6.19) gives a solution of (6.9) and hence \( Y_1 \) is a complex solution of (6.9) as it is a part of (6.19).
System (6.9) in a matrix form can be written as:
\[
\frac{d\mathbf{X}}{dt} = A\mathbf{X}
\]  
(6.23)

As \(Y_1\) is a solution, it satisfies (6.23):
\[
\frac{dY_1}{dt} = AY_1
\]
\[
\frac{dy_1}{dt} + i\frac{dy_2}{dt} = A(y_1 + iy_2)
\]
\[
\frac{dy_1}{dt} + i\frac{dy_2}{dt} = Ay_1 + iAy_2
\]

Equating the real and imaginary parts yields
\[
\frac{dy_1}{dt} = Ay_1 \quad \frac{dy_2}{dt} = Ay_2
\]

Hence \(y_1\) and \(y_2\) are real solutions of (6.9). Finally, because \(y_1\) and \(y_2\) are real solutions of (6.9) the general solution is given by the formula:
\[
\begin{pmatrix} x \\ y \end{pmatrix} = C_1y_1 + C_2y_2
\]  
(6.24)

where \(C_1\) and \(C_2\) are arbitrary constants and \(y_1\) and \(y_2\) are given by (6.23).

**Example** Find the general solution of the following system.
\[
\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or} \quad \begin{cases} \frac{dx}{dt} = 2y \\ \frac{dy}{dt} = -2x \end{cases}
\]  
(6.25)

**Solution.** The eigen values are given by: \(\text{Det} \begin{vmatrix} -\lambda & 2 \\ -2 & -\lambda \end{vmatrix} = \lambda^2 + 4 = 0, \quad \lambda_{1,2} = \pm\sqrt{-4} = \pm2i\). So the eigen vector \(v_1\) corresponding to the eigen value \(\lambda = 2i\) is:
\[
v_1 = \begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} = \begin{pmatrix} -2 \\ 0 - 2i \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ -2 \end{pmatrix} = v_r + iv_i
\]  
(6.26)

So,
\[
y_1 = e^{0\left(\begin{pmatrix} -2 \\ 0 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ -2 \end{pmatrix} \sin 2t\right)} = \begin{pmatrix} -2\cos 2t \\ 2\sin 2t \end{pmatrix}
\]
\[
y_2 = e^{0\left(\begin{pmatrix} -2 \\ 0 \end{pmatrix} \sin 2t + \begin{pmatrix} 0 \\ -2 \end{pmatrix} \cos 2t\right)} = \begin{pmatrix} -2\sin 2t \\ -2\cos 2t \end{pmatrix}
\]

Therefore the solution from the formulae (6.24) is:
\[
\begin{pmatrix} x \\ y \end{pmatrix} = C_1\begin{pmatrix} -2\cos 2t \\ 2\sin 2t \end{pmatrix} + C_2\begin{pmatrix} -2\sin 2t \\ -2\cos 2t \end{pmatrix}
\]  
(6.27)

Or if we denote \(-2C_1 = A\) and \(-2C_2 = B\) we get:
\[
\begin{cases} x = A\cos 2t + B\sin 2t \\ y = -A\sin 2t + B\cos 2t \end{cases}
\]  
(6.28)
6.4.1 Canonical or Jordan normal form

If $\lambda_{1,2} = \alpha \pm i\beta$ are the complex eigen values of the matrix $A$ and $v_r$ corresponds to the real part of the complex eigen vector and $v_i$ corresponds to the imaginary part of the complex eigen vector, then the matrix $T$ which columns are the vectors $v_r$ and $v_i$ transforms the matrix $A$ into the canonical form $J$ by the transformation:

$$J = T^{-1}AT = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$ (6.29)

In the canonical coordinate system given by transformation $T$ the differential equations will be:

$$\frac{du}{dt} = \alpha u - \beta v$$
$$\frac{dv}{dt} = \beta u + \alpha v$$

or,

$$\frac{dz}{dt} = \lambda z$$

where $z = x + iy$, $\lambda = \alpha + i\beta$

6.5 Equilibria types for complex eigen values

Conclusion 5 If the eigen values of the system are $\lambda_{12} = \pm i\beta$, we have an equilibrium point called a center. The dynamics of variables $x,y$ are oscillations. The phase portrait is a set of embedded ellipses.

Conclusion 6 The imaginary part of the eigen values results in the rotation of a trajectory on a phase portrait.

Conclusion 7 If the eigen values of the system are $\lambda_{12} = \alpha \pm i\beta; \alpha < 0$, we have an equilibrium point called a stable spiral.

Conclusion 8 If the eigen values of the system are $\lambda_{12} = \alpha \pm i\beta; \alpha > 0$, we have an equilibrium point called a non-stable spiral.

Figure 6.4: A center point (a), a stable spiral (b) and a non-stable spiral (c)
6.6 Stability of equilibrium

We will call an equilibrium point **stable**, if there is a neighborhood of this equilibrium, such that all trajectories which start at this neighborhood will converge to the equilibrium. We will call the equilibrium point **non-stable**, if there is at least one diverging trajectory from each close neighborhood of this equilibrium.

![Diagram of stable and non-stable equilibria](image)

**Figure 6.5: Phase portrait around a stable (a) and a non-stable (b) equilibria**

**Stable equilibria**

1. Stable node $\lambda_1 < 0; \lambda_2 < 0$ real
2. Stable spiral $\lambda_{12} = \alpha \pm i\beta; \alpha < 0$

**Non-stable equilibria**

1. Non-stable node $\lambda_1 > 0; \lambda_2 > 0$ real
2. Non-stable spiral $\lambda_{12} = \alpha \pm i\beta; \alpha > 0$
3. Saddle point $\lambda_1 < 0; \lambda_2 > 0$; or $\lambda_1 > 0; \lambda_2 < 0$ real

**Theorem 5** If all eigenvalues of the linear system (6.9) have negative real parts, then the equilibrium point $x = 0, y = 0$ is stable.

6.7 Express method for finding type of equilibrium

**Definition 12** The trace of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\text{tr}A = a + d$.

The determinant of the matrix $A$ is $\det A = ad - cb$. 

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42
Figure 6.6: Express method for finding type of equilibrium
Chapter 7

Hopf bifurcation

Consider a planar ODE:
\[
\begin{align*}
\dot{x} &= f(x,y) \\
\dot{y} &= g(x,y)
\end{align*}
\]  
(7.1)

Equilibria points of this ODE are given by the conditions:
\[
\begin{align*}
f(x^*,y^*) &= 0 \\
g(x^*,y^*) &= 0
\end{align*}
\]  
(7.2)

We can linearize our ODE close to equilibria using Taylor series. For that we first shift the equilibrium to the point \((0,0)\) We find the following:
\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix}
= J
\begin{pmatrix}
x \\
y
\end{pmatrix}
\]  
(7.3)

where \(J\) is the matrix called the \text{Jacobian}

\[
J = \begin{pmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{pmatrix}
\]  
(7.4)

The Jacobian has two eigen values. The stability of equilibrium is determined by eigen values of the Jacobian.

**Theorem 6** Assume that linearization (7.3) of ODE (7.1) has two eigen values which can be both real \(\lambda_1, \lambda_2\), or complex conjugate \(\lambda_{1,2} = \alpha \pm i\beta\). The equilibrium point of the ODE (7.1) is stable if all eigen values of the Jacobian have negative real parts.

As in previous cases stability is closely connected with hyperbolicity:

**Definition 13** A equilibrium point of the ODE (7.1) is said to be hyperbolic if all the eigen value of the Jacobian matrix have nonzero real parts.

And again in hyperbolic case linear and non-linear systems are equivalent to each other.

**Theorem 7** Let \(x^* = 0, y^* = 0\) be a hyperbolic equilibrium point of the ODE (7.1). Then there is a neighborhood this equilibrium where the original system (7.1) is equivalent to its linear ODE (7.3).

44
Note, that because any equilibrium can be shifted to the point $x^* = 0, y^* = 0$, the above theorem is valid for any equilibrium.

Similar theorem exists for the system with parameter:

$$\begin{align*}
\dot{x} &= f(x, y, c) \\
\dot{y} &= g(x, y, c)
\end{align*}$$

Therefore we do not expect any bifurcations close to hyperbolic equilibria. Let us study non-hyperbolic cases. Here we consider the case when:

$$\lambda_{1,2}(c) = \alpha(c) \pm i\beta(c); \quad \alpha(0) = 0$$

### 7.1 Normalization

Assume, that a system of two ODEs (7.5) has an equilibrium point at $x = 0, y = 0$ for all $c$ close to $(c = 0)$, and the eigen values for this equilibrium have a zero real part at $c = 0$:

$$\begin{align*}
\lambda_{1,2}(c) &= 0; \quad g(0, 0, c) = 0; \\
\lambda_{1,2}(c) &= \alpha(c) \pm i\beta(c); \quad \alpha(0) = 0
\end{align*}$$

Let us study behavior of our system around this point. Our first step will be simplification of our system using Maclaurin series.

#### 7.1.1 Linear terms.

Let us expand right hand sides of (7.5) into the Maclaurin series. In order to simplify computations we will keep $c$ as a parameter, and write a Maclaurin series for the functions $f(x, y, c), g(x, y, c)$, as the function of two variables $(x, y)$. We get:

$$\begin{align*}
\dot{x} &= f(0, 0, c) + x\frac{\partial f}{\partial x}(0, 0) + y\frac{\partial f}{\partial y}(0, 0) + \frac{x^2}{2}\frac{\partial^2 f}{\partial x^2}(0, 0) + xy\frac{\partial^2 f}{\partial x\partial y}(0, 0) + \frac{y^2}{2}\frac{\partial^2 f}{\partial y^2}(0, 0) + \cdots \\
\dot{y} &= g(0, 0, c) + x\frac{\partial g}{\partial x}(0, 0) + y\frac{\partial g}{\partial y}(0, 0) + \frac{x^2}{2}\frac{\partial^2 g}{\partial x^2}(0, 0) + xy\frac{\partial^2 g}{\partial x\partial y}(0, 0) + \frac{y^2}{2}\frac{\partial^2 g}{\partial y^2}(0, 0) + \cdots
\end{align*}$$

Note also, that because we keep $c$ as a parameter, the coefficients $(f(0, 0, c), \frac{\partial f}{\partial x}, \text{etc.})$ are not constants, but some functions of $c$.

Because $(0, 0)$ is an equilibrium $f(0, 0, c) = 0; \quad g(0, 0, c) = 0; \quad (\text{see (7.7)}))$. Let us denote all terms of the second and higher order in the first equation as $f2(x, y, c)$ and all similar terms in the second equation as $g2(x, y, c)$:

$$\begin{align*}
\dot{x} &= x\frac{\partial f}{\partial x}(0, 0) + y\frac{\partial f}{\partial y}(0, 0) + f2(x, y, c) \\
\dot{y} &= x\frac{\partial g}{\partial x}(0, 0) + y\frac{\partial g}{\partial y}(0, 0) + g2(x, y, c)
\end{align*}$$

Now, we will start simplification of system (7.9). Our first step will include transformation of the linear terms. For that we note, that the matrix $J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$ which gives linear terms of our system can be transformed to the Jordan normal form using the procedure described in section 6.4.1. For that we need to find eigen values and one eigen vector of the matrix $(v_r + iv_i)$, and then use the transformation by the matrix $T$ which columns are the vectors $v_r$ and $v_i$. This will transform the system of ODE into the canonical form with the matrix $D$ transformation:
7.1.3 Removing of one nonlinear term

Let us illustrate the procedure of removing the terms from (7.14) on example of one term.

Expressed as:

\[ F \]

Following quadratic change of variables:

\[ \frac{du}{dt} = \alpha(c)u - \beta(c)v + F_2(u,v,c) \]
\[ \frac{dv}{dt} = \beta(c)u + \alpha(c)v + G_2(u,v,c) \]

where \( F_2, G_2 \) are the functions \( f_2, g_2 \) after the transformation \( T \).

Now, let us introduce the complex variables \( z = u + iv, \bar{z} = u - iv \), and therefore \( x, y \) can be expressed as:

\[ u = \frac{z + \bar{z}}{2}, \quad v = \frac{z - \bar{z}}{2i} \]

If we multiply the second equation in (7.11) by \( i \) and add it to the first equation we get:

\[ \dot{z} = \lambda z + F(z, \bar{z}, c) \]  

(7.12)

where \( F(z, \bar{z}, c) \) is a function which obtained during this transformation form the functions \( F_2(u,v,c), G_2(u,v,c) \) after replacing \( u, v \) by \( z, \bar{z} \). Note, that \( F(z, \bar{z}, c) \) contains terms of the order of \( z^2, \bar{z}^2 \).

Note, that because \( z \) and \( \bar{z} \) are very similar to each other, \( z = x + iy, \bar{z} = x - iy \), we will not distinguish between \( O(z^n) \) and \( O(\bar{z}^n) \), and will denote the both cases as \( O(|z|^n) \).

7.1.2 Maclaurin approximation

Let us write a Maclaurin series for the function \( F(z, \bar{z}, c) \), as the function of two variables \( (z, \bar{z}) \), keeping in mind that it does not contain any linear and the first order terms:

\[ F(z, \bar{z}, c) = \frac{\partial^2 F}{\partial z^2} \frac{z^2}{2} + \frac{\partial^2 F}{\partial z \partial \bar{z}} z \bar{z} + \frac{\partial^2 F}{\partial \bar{z}^2} \frac{\bar{z}^2}{2} \]
\[ + \frac{\partial^3 F}{\partial z^3} \frac{z^3}{6} + \frac{\partial^3 F}{\partial z^2 \partial \bar{z}} \frac{z^2 \bar{z}}{2} + \frac{\partial^3 F}{\partial z \partial \bar{z}^2} \frac{z \bar{z}^2}{2} + \frac{\partial^3 F}{\partial \bar{z}^3} \frac{\bar{z}^3}{6} \ldots \]

(7.13)

Note, that because we keep \( c \) as a parameter, the coefficients in this series are not constants, but some functions of \( c \). Finally, our approximation for ODE (7.12) will be

\[ \dot{z} = \lambda z + \frac{\partial^2 F}{\partial z^2} \frac{z^2}{2} + \frac{\partial^2 F}{\partial z \partial \bar{z}} z \bar{z} + \frac{\partial^2 F}{\partial \bar{z}^2} \frac{\bar{z}^2}{2} \]
\[ + \frac{\partial^3 F}{\partial z^3} \frac{z^3}{6} + \frac{\partial^3 F}{\partial z^2 \partial \bar{z}} \frac{z^2 \bar{z}}{2} + \frac{\partial^3 F}{\partial z \partial \bar{z}^2} \frac{z \bar{z}^2}{2} + \frac{\partial^3 F}{\partial \bar{z}^3} \frac{\bar{z}^3}{6} \ldots \]

(7.14)

Our next step will be removing of nonlinear terms from equation (7.14).

7.1.3 Removing of one nonlinear term

Let us illustrate the procedure of removing the terms from (7.14) on example of one term \( \frac{\partial^2 F}{\partial z^2} \frac{z^2}{2} \).

We assume that our equation is

\[ \dot{z} = \lambda z + Az^2 \]

(7.15)

here \( A = \frac{1}{2} \frac{\partial^2 F}{\partial z^2} \), \( A \) is some function of \( c \). It turns out that we can remove this term by the following quadratic change of variables:

\[ z = w + aw^2 \]

(7.16)
Note, that because we want to study our system close to the equilibrium point \( z = 0 \) we need a change of variables which work good for small values of \( z \).

The first step is to find the inverse change of variables, i.e., \( w \) as a function of \( z \). We can find it, by rewriting (7.16) as a quadratic equation and by finding its roots. (7.16) is equivalent to

\[
aw^2 + w - z = 0 \quad \text{hence} \quad w_{1,2} = \frac{-1 \pm \sqrt{1 + 4az}}{2a} \quad (7.17)
\]

Why do we have two roots here? We can easily see it from the graph of the function \( z = w + aw^2 \) shown in fig.7.1. The graph of this function is a parabola open upwards with two roots \( z = 0, z = -a \). Finding of the inverse transformation means finding value of \( w \) if we know the value of \( z \). We see in fig.7.1 that for many \( z \) we indeed have two possible values of \( w \).

![Figure 7.1: Schematic graph of \( z = w + aw^2 \)](image)

We need to choose one branch of this parabola. It is quite obvious that we need the right branch, because it goes through the point \( z = 0, w = 0 \), and therefore it maps our equilibrium point \( z = 0 \) to the point \( w = 0 \). The right branch is given by the + sign in equation (7.17) and therefore the inverse transformation is

\[
w = \frac{-1 + \sqrt{1 + 4az}}{2a} \quad (7.18)
\]

We are interested in small values of \( z \) only, therefore we can further simplify expression (7.18) using the following Maclaurin series:

\[
\sqrt{1 + x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 \ldots \quad (7.19)
\]

Therefore, for we get

\[
\sqrt{1 + 4az} \approx 1 + 2az - 2a^2z^2 + 4a^3z^3 + \ldots
\]

After substitution this expression into (7.18) we get:

\[
w \approx \frac{-1 + 1 + 2az - 2a^2z^2 + 4a^3z^3}{2a} = -az^2 + 2a^2z^3 
\]

(7.20)

Because our aim is to remove the term which is quadratic in \( z \), it is sufficient to keep only quadratic terms in the inverse transformation (7.20):

\[
w \approx z - az^2 
\]

(7.21)

Now we are ready to start the removing of the nonlinear term \( Az^2 \).

The first step is finding dynamics of the new variable \( w \):

\[
\dot{w} = \dot{z} - 2azz 
\]

(7.22)
Now, we have to replace $\dot{z}$ by its expression (7.15), we find

$$\dot{w} = \lambda z + A z^2 - 2a z (\lambda z + A z^2) = \lambda z + (A - 2\lambda a) z^2 - a A z^3$$

(7.23)

Because we are not interested in terms of the 3rd or higher order in $z$ let us denote them as $O(|z|^3)$:

$$\dot{w} = \lambda z + (A - 2\lambda a) z^2 + O(|z|^3)$$

(7.24)

Now we need to replace $z$ using the direct transformation (7.16):

$$\dot{w} = \lambda (w + a w^2) + (A - 2\lambda a) (w + a w^2)^2 + O(|w|^3) = \lambda w + \lambda a w^2 + (A - 2\lambda a)(w^2 + 2a w^3 + a^2 w^4) + O(|w|^3)$$

Note, that $O(|z|^3)$ naturally becomes $O(|w|^3)$ as any term of the third order in $z^3$ will be obviously of the third order in $w$ as $(w + a w^2)^3 = O(|w|^3)$. Note also, that we can put other terms like $2aw^3, a^2 w^4$ to $O(|w|^3)$ yielding:

$$\dot{w} = \lambda w + w^2 (\lambda a + A - 2\lambda a) + O(|w|^3) = \lambda w + w^2 (A - \lambda a) + O(|w|^3)$$

Therefore if we choose

$$a = \frac{A}{\lambda}$$

then our transformation (7.16) will remove the quadratic term in $\dot{z} = \lambda z + A z^2$. Of course this transformation can change cubic terms which are collected in $O(|w|^3)$. However, they are of the higher order and we can handle them latter.

### 7.1.4 Removing of other nonlinear term

We can use similar procedure and remove other nonlinear terms in the Maclaurin decomposition of our nonlinear function (7.14). However the procedure or their removing although similar, but it involves a bit more complex mathematics and some extra knowledge on complex numbers. Therefore, you can skip the most of this section if you are not interested in these details and go directly to the last paragraph of this section.

In the previous section we have removed the term $A z^2$, now let us remove the term which involves the other variable $\bar{z}$.

For example let us remove the term:

$$\dot{\bar{z}} = \lambda \bar{z} + B \bar{z}^2$$

(7.25)

where $B = \frac{1}{2} \frac{\partial^2 F}{\partial \bar{z}^2}$

In this case we can also achieve it by the transformation:

$$z = w + b \bar{w}^2$$

(7.26)

We will also need a complex conjugate of the direct transformation (7.26):

$$\bar{z} = \bar{w} + \bar{b} w^2 \quad as \quad \bar{w} = w$$

(7.27)

However, finding of the transformation inverse to (7.26) is more difficult. We will present here some basic ideas how this inverse transformation can be found.

We can rewrite (7.26) as

$$w = z - b \bar{w}^2$$

(7.28)
so we see that in order to find an inverse transformation we need to express \( \bar{w} \) in terms of \( z, \bar{z} \). We also know from the previous analysis that we need this inverse transformation up to the second order only.

In order to express \( \bar{w} \) in terms of \( z, \bar{z} \) we take a complex conjugate from the both sides of equation (7.28).

\[
\bar{w} = \bar{z} - \bar{b}w^2 \quad \text{as} \quad \bar{w} = w
\]  

(7.29)

Important step here is to note, that the term \( \bar{b}w^2 \) is of the second order in \( w \) or it can be denoted as \( O(|w|^2) \). It is also important to note, that if a term is \( O(|w|^2) \) it will be also of the second order in \( z, \bar{z} \). This is because as we see from (7.28) \( w \approx z \). Thus, we can rewrite (7.28) as:

\[
\bar{w} = \bar{z} + O(|z|^2)
\]

(7.30)

And after substitution of (7.30) into (7.28) we get

\[
w = z - b(\bar{z} + O(|z|^2))^2 = z - b\bar{z}^2 + O(|z|^3)
\]

(7.31)

This is because \( (\bar{z} + O(|z|^2))^2 = \bar{z}^2 + 2\bar{z}(O(|z|^2) + (O(|z|^2)^2 \text{ and } \bar{z}O(|z|^2) = O(|z|^3), \text{ and }

O(|z|^2)^2 = O(|z|^4).

Thus the inverse transformation up to the second order is:

\[
w = z - b\bar{z}^2
\]

(7.32)

The next problem is that at some stage we will need to find the time derivative \( \dot{\bar{z}} \). It can be found from (7.25) by taking a complex conjugate of the whole equation, and we will need just linear terms of this formulae:

\[
\dot{\bar{z}} = \bar{\lambda}\bar{z} + B\bar{z}^2 = \bar{\lambda}\bar{z} + O(|z|^2)
\]

(7.33)

Now, let us follow the same plan as in the previous section. We will find:

Dynamics of the new variable \( w \):

\[
\dot{w} = \dot{z} - 2b\bar{z}\ddot{z}
\]

(7.34)

Now, we have to replace \( \dot{z} \) by its expression (7.25), and \( \ddot{z} \) by (7.33). We find

\[
\dot{w} = \lambda z + B\bar{z}^2 - 2b\bar{z}(\bar{\lambda}\bar{z} + O(|z|^2)) = \lambda z + (B - 2\bar{\lambda}b)\bar{z}^2 + O(|z|^3)
\]

(7.35)

Now we need to replace \( z \) using transformation (7.26) and \( \bar{z} \) using transformation (7.27):

\[
\dot{w} = \lambda(w + \bar{b}w^2) + (B - 2\bar{\lambda}b)(\bar{w} + \bar{b}w^2)^2 = \lambda w + (\bar{\lambda}b + B - 2\bar{\lambda}b)\bar{w}^2 + O(|w|^3)
\]

This is because \( (\bar{w} + \bar{b}w^2)^2 = \bar{w}^2 + O(|w|^3) \)

Finally we get

\[
\dot{w} = \lambda w + (\bar{\lambda}b + B - 2\bar{\lambda}b)\bar{w}^2 + O(|w|^3)
\]

Therefore if we choose:

\[
b = \frac{B}{2\bar{\lambda} - \lambda}
\]

we can remove the nonlinear term in \( \dot{z} = \lambda z + B\bar{z}^2 \).
Similarly we can remove the term \( \dot{z} = \lambda z + Cz \bar{z} \) using the transformation

\[
z = w + cw \bar{w}; \quad \text{with } c = \frac{C}{\lambda}
\]

Thus we were able to remove all the quadratic terms in (7.14). These substitutions are correct because the denominators are nonzero for all sufficiently small \( c \) since \( \lambda(0) = i\beta(0) \) with \( \beta(0) > 0 \). After all these transformation our ODE will have only cubic terms:

\[
\dot{z} = \lambda z + \frac{\partial^3 F}{\partial z^3} z^3 + \frac{\partial^3 F}{\partial z^2 \partial \bar{z}} z^2 \bar{z} + \frac{\partial^3 F}{\partial \bar{z}^3} \bar{z}^3 \ldots
\]

(7.36)

We can similarly remove almost all cubic terms:

\[
\dot{z} = \lambda z + Dz^3 \quad \text{by} \quad z = w + dw^3 \quad \text{with} \quad d = \frac{D}{2\lambda}
\]

\[
\dot{z} = \lambda z + Fz^2 \bar{z} \quad \text{by} \quad z = w + f w^2 \bar{w} \quad \text{with} \quad f = \frac{F}{2\lambda}
\]

\[
\dot{z} = \lambda z + Gz^3 \quad \text{by} \quad z = w + g \bar{w}^3 \quad \text{with} \quad g = \frac{G}{3\lambda - \lambda}
\]

(7.37)

However there will be one non-removable term. If we would like to remove in a similar way the term:

\[
\dot{z} = \lambda z + Ez^2 \bar{z} \quad \text{by} \quad z = w + ew^2 \bar{w}
\]

We will find that after such transformation our equation becomes:

\[
\dot{w} = \lambda w + w^2 \bar{w}(E - e(\lambda + \bar{\lambda}))
\]

However, because at the bifurcation point \( c = 0, \lambda(0) + \bar{\lambda}(0) = i\beta(0) - i\beta(0) = 0, e(\lambda + \bar{\lambda}) = 0 \) for any \( e \), and the term \( w^2 \bar{w}(E - e(\lambda + \bar{\lambda}) = Ew^2 \bar{w} \) persists for any value of \( e \), i.e., it is not removable.

Our final conclusion is, that any system (7.14) can be transformed to the form:

\[
\dot{w} = \lambda w + \delta w^2 \bar{w}
\]

(7.38)

### 7.1.5 Back to real numbers

We got a normal form for a complex representation of our system. Now, let us recall, that our complex variable \( w \) is a combination of two real variables \( w = u + iv \), and our aim is to find bifurcation of the phase portrait on the \( u, v \) plane. It turns out, that because our phase portrait will include circles, spirals, etc, it is more convenient to write it in a polar coordinate system \( \rho, \phi \), where \( u = \rho \cos \phi; v = \rho \sin \phi; \) or for a complex variable \( w = u + iv = \rho(\cos \phi + i\sin \phi) = \rho e^{i\phi} \). We need to write from one ODE (7.38) a system of two ODEs from which we can find \( \rho \) and \( \phi \) as a functions of time. Note, that \( \lambda = \alpha + i\beta \). Also note, that \( \delta \) is a complex number, which we can rewrite as \( \delta = l + id \). Equation (7.38) in such notations becomes:

\[
\dot{w} = (\alpha + i\beta)w + (l + id)w^2 \bar{w}
\]

(7.39)

We can easily find that

\[
\dot{\rho} e^{i\phi} + i\dot{\phi} \rho e^{i\phi}
\]

50
And that
\[ w^2 \tilde{w} = \delta p^3 e^{i\phi} \text{ as } \tilde{w} = pe^{-i\phi} \]

Therefore equation (7.38) in polar representation becomes:
\[ \dot{p} e^{i\phi} + i\dot{\phi} p e^{i\phi} = (\alpha + i\beta)p + (l + id)p^3 e^{i\phi} \]

We can cancel \( e^{i\phi} \) and get:
\[ \dot{p} + i\dot{\phi} p = (\alpha + i\beta)p + (l + id)p^3 \]  \hspace{1cm} (7.40)

By equating the real and imaginary parts of equation (7.40) we get:
\[ \dot{\rho} = \alpha \rho + lp^3 \]
\[ \dot{\phi} = \beta + dp^2 \]  \hspace{1cm} (7.41)

If \( \beta \neq 0 \) we can neglect \( p^2 \) compared to \( \beta \) and our equation becomes:
\[ \dot{\rho} = \alpha \rho + lp^3 \]
\[ \dot{\phi} = \beta \]  \hspace{1cm} (7.42)

In equation (7.42) \( \alpha \) is a function of \( c \). If \( \frac{\partial \alpha}{\partial c} \neq 0 \) we can introduce a new parameter \( \pm \mu \) as we did in previous normalizations. We will use the ‘+’ sign if \( \frac{\partial \alpha}{\partial c} > 0 \) the ‘-’ sign if \( \frac{\partial \alpha}{\partial c} < 0 \)
After that our equation becomes:
\[ \dot{\rho} = \pm \mu \rho + lp^3 \]
\[ \dot{\phi} = \beta \]  \hspace{1cm} (7.43)

By rescaling the amplitude of \( \rho \) this equation can be transformed into:
\[ \dot{r} = \pm \gamma r \pm r^3 \]
\[ \dot{\theta} = 1 \]  \hspace{1cm} (7.44)

where the sign at \( \pm r^3 \) comes from the sign of \( l \).

### 7.2 Study of the normal form

As \( \dot{r} = \pm \gamma r \pm r^3 \) is the normal form form the pitch fork bifurcation, then the bifurcation diagrams for \( r \) will be the same as for the pitch fork bifurcation. The second equation \( \dot{\theta} = 1 \) will give us rotation of a trajectory.

We have the bifurcation diagrams for \( r \), and the dynamics in 2D which are shown in the figures at the end of this chapter.

### 7.3 Theorem. Hopf bifurcation.

Let a 2D ODE
\[ \dot{x} = f(x,y,c) \]
\[ \dot{y} = g(x,y,c) \]  \hspace{1cm} (7.45)

has a non-hyperbolic equilibrium point at \( x = 0, y = 0, c = 0 \), such that eigen values of the Jacobian matrix are:
\[ \lambda = \alpha(c) \pm i\beta(c); \quad \alpha(0) = 0 \]  \hspace{1cm} (7.46)
then if
\[ \frac{\partial \alpha}{\partial c}(0) \neq 0; \quad \text{and} \quad \beta(0) \neq 0; \quad \text{and} \quad \text{Re}(c_1) \neq 0; \] (7.47)

there is a coordinate change which transforms (7.45) into the following form:
\[ \dot{r} = \pm \gamma r \pm r^3 \]
\[ \dot{\theta} = 1 \] (7.48)

and the Hopf bifurcation takes place.

1. Note that the sign at \( \pm \gamma \) is the same as the sign of \( \frac{\partial \alpha}{\partial c}(0) \) and the sign of \( \pm r^3 \) is the same as the sign of \( \text{Re}(c_1) \).

2. Note, that stability of equilibrium at bifurcation point determines the stability of the new born limit cycle, i.e., if the equilibrium at \( \gamma = 0 \) is stable, than the limit cycle is also stable and vise versa.

3. Note, that the cases, when we have stable limit cycle are called supercritical Hopf bifurcation, the cases when we have unstable limit cycle are called subcritical Hopf bifurcation.

4. Note, that this normal form has the following representation in the Cartesian coordinate system:
\[ \begin{cases} 
\frac{dx}{dt} = -y + x[\pm \gamma \pm (x^2 + y^2)] \\
\frac{dy}{dt} = x + y[\pm \gamma \pm (x^2 + y^2)] 
\end{cases} \] (7.49)

### 7.4 Stability index \( \text{Re}(c_1) \)

For the system
\[ \begin{cases} 
\frac{dx}{dt} = A + \omega y + Y^1 \\
\frac{dy}{dt} = B - \omega x + Y^2 
\end{cases} \] (7.50)

the stability index \( I = \text{Re}(c_1) \) is:
\[ I = \omega (Y^1_{xx} + Y^1_{xy} + Y^2_{xy} + Y^2_{yy}) + (Y^1_{xx} Y^1_{xy} + Y^2_{xx} Y^2_{xy}) \\
+ (Y^1_{xy} Y^1_{yy} - Y^1_{yy} Y^1_{xy} - Y^1_{xx} Y^2_{xy}) \]

where
\[ Y^1_{xy} = \frac{\partial^2 Y^1}{\partial x \partial y}(0,0) \quad Y^2_{xxy} = \frac{\partial^3 Y^2}{\partial x \partial x \partial y}(0,0), \quad \text{etc} \]

For the system
\[ \begin{cases} 
\frac{dx}{dt} = A - \omega y + Y^1 \\
\frac{dy}{dt} = B + \omega x + Y^2 
\end{cases} \] (7.51)

the stability index is:
\[ I = \omega (Y_{xxx}^1 + Y_{xxy}^1 + Y_{xxy}^2 + Y_{yyy}^2) \]
\[ + (-Y_{xxx}^1 y_2 + Y_{xxy}^1 y_1 - Y_{xxy}^2 y_2) \]
\[ - Y_{yyy}^2 y_2 + Y_{yyy}^1 y_1 + Y_{yyy}^1 y_2) \]

If the index \( I \) is negative, then the origin is stable.

**Bifurcation diagrams**

\[ \frac{dr}{dt} = \gamma r - r^3 \]

\[ \frac{dr}{dt} = -\gamma r + r^3 \]
\[ \frac{dr}{dt} = \gamma r + r^3 \]

\[ \frac{dr}{dt} = -\gamma r - r^3 \]
Chapter 8

Center manifold theory

8.1 Main theorems

Consider a non-linear system of two ODEs. Assume that this system has an equilibrium and assume that one eigen value of the Jacobian at this equilibrium is $\lambda_1 = 0$. Then, in a canonical form our system will be the following:

\[
\begin{align*}
\frac{dx}{dt} &= f(x,y) \\
\frac{dy}{dt} &= \lambda_2 y + g(x,y)
\end{align*}
\] (8.1)

Here $f, g$ are the functions of the second order in $x$ and $y$, i.e., they do not have any linear terms in their Taylor series.

If we consider a linearization of system (8.1) we get a system:

\[
\begin{align*}
\frac{dx}{dt} &= 0 \\
\frac{dy}{dt} &= \lambda_2 y
\end{align*}
\] (8.2)

System (8.2) has the following phase portrait which includes a central manifold $E^c$ and $E^s$ a stable (unstable) manifold $E^s$:

![Phase portrait](image)

Figure 8.1: The center ($E^c$) and stable ($E^s$) manifolds in a linear system with $\lambda_1 = 0, \lambda_2 < 0$ in a canonical form (8.2)

It turns out that we also have similar manifolds in a non-linear system (8.1).

**Theorem 8** 1. Let $x = 0, y = 0$ is an equilibrium point of the system

\[
\begin{align*}
\frac{dx}{dt} &= F(x,y) \\
\frac{dy}{dt} &= G(x,y)
\end{align*}
\] (8.3)
Let a linearization of this system has one eigen value $\lambda_1 = 0$ and the other $\lambda_2 \neq 0$. Hence the linear system has a central manifold $E^c$ and stable (non-stable) manifold $E^u$.

Then there exist central $W^c$ and stable $W^c$ (non-stable $W^u$) manifolds in a non-linear system (8.3) which are tangent to $E^c$ and $E^s, E^u$, respectively.

2. If our system in a canonical form:

\[
\begin{align*}
\frac{dx}{dt} &= f(x, y) \\
\frac{dy}{dt} &= \lambda_2 y + g(x, y)
\end{align*}
\]

then the center manifold can be written as:

\[
y = h^c(x), \quad h^c(0) = 0; \quad \frac{dh^c}{dx}(0) = 0
\]

and the flow on the center manifold is:

\[
\frac{dx}{dt} = f(x, h^c(x))
\]

The manifolds are shown schematically in fig.2.

![Diagram showing center and stable manifolds in a nonlinear system](image)

Figure 8.2: The center ($W^c$) and stable ($W^s$) manifolds in a nonlinear system

We clearly see the differences between the manifold of linear system shown in fig.1 and manifolds of non-linear system shown in fig.2: the manifolds in non-linear system are not straight lines. This because the vector field of system (8.4) depends on nonlinear functions $f(x, y), g(x, y)$ and therefore it in general is not parallel to the $x$ of $y$ axis. However the manifolds in non-linear system are tangent to the manifolds in linear system at the origin.

Note, that all the manifolds which we discussed above, are so-called invariant manifolds. This means, that if the initial conditions for a trajectory are located at the manifold, then the whole trajectory will be also a part of this manifold. This property of the center manifold is used in the following plan in order to compute it.

8.2 Plan for computation of center manifold

**Step1** Put system into the canonical form (8.1). You should transform linear as well as non-linear terms.
Step 2 Find equation for center manifold and add it to the system in the following way

\[
\begin{align*}
\frac{dx}{dt} &= f(x, y) \\
\frac{dy}{dt} &= \lambda_2 y + g(x, y) \\
y &= h^c(x)
\end{align*}
\] (8.7)

Step 3 Solve system (8.7) using Taylor expansion and find Taylor expansion for \( y = h^c(x) \)

Step 4 Find flow on center manifold as

\[
\frac{dx}{dt} = f(x, h^c(x))
\]

1. Note, that system (8.7) informally means the following: the first and the second equations give us change of \( x \) and \( y \) in the course of time, the third equation requires that our \( x \) and \( y \) are always on some line \( y = h^c(x) \). Therefore it means, that we require that \( x \) and \( y \) while changing in the course of time, should always be on the line \( y = h^c(x) \). Therefore this line will be an invariant manifold, i.e., if the initial conditions for a trajectory are on this line \( y = h^c(x) \), the whole trajectory will be also on this line.

2. Note, that for the system with \( \lambda_2 = 0, \lambda_1 \neq 0 \) the equations similar to (8.7) will be:

\[
\begin{align*}
\frac{dx}{dt} &= \lambda_1 x + f(x, y) \\
\frac{dy}{dt} &= g(x, y) \\
x &= h^c(y)
\end{align*}
\] (8.8)

8.3 System with a parameter

Now let us consider a system with one parameter:

\[
\begin{align*}
\frac{dx}{dt} &= f(x, y, c) \\
\frac{dy}{dt} &= g(x, y, c)
\end{align*}
\] (8.9)

Assume that this system has a non-hyperbolic equilibrium at \( c = 0 \), i.e., it has one eigenvalue which is \( \lambda_1 = 0 \). What will be behavior of this system at close values of \( c \)?

We know how to study system (8.9) at \( c = 0 \), i.e., we need to find center manifold and the flow of our system on this center manifold.

More formally: we need to transform our system to the canonical form at \( c = 0 \):

\[
\begin{align*}
\frac{dx}{dt} &= f(x, y, 0) \\
\frac{dy}{dt} &= \lambda_2 y + g(x, y, 0)
\end{align*}
\] (8.10)

Find \( h^c(x) \), substitute it to the first equation:

\[
\frac{dx}{dt} = f(x, h^c(x), 0)
\]

From this equation we find the flow along the center manifold. Then we draw this flow on a phase portrait and add a flow along the \( y \) axis, which is converging at \( \lambda_2 < 0 \), or diverging at \( \lambda_2 > 0 \).

It turns out that we can use a similar plan for systems with parameter.
First we will transform our system to the form:

\[
\begin{align*}
\frac{dx}{dt} &= F(x, y, c) \\
\frac{dy}{dt} &= \lambda_2(c)y + G(x, y, c)
\end{align*}
\] (8.11)

Note, that the functions \(F(x, y, c), G(x, y, c)\) are at \(c = 0\) of the second order for \(x\) and \(y\), i.e., all first order partial derivatives of \(F(x, y, c)\) with respect to \(x\) and \(y\) are zeros

\[
\frac{\partial F}{\partial x}(0, 0, 0) = 0; \quad \frac{\partial F}{\partial y}(0, 0, 0) = 0; \quad \frac{\partial G}{\partial x}(0, 0, 0) = 0; \quad \frac{\partial G}{\partial y}(0, 0, 0) = 0;
\] (8.12)

It is possible to prove, that we can find kind of ”center manifold” for system with a parameter (8.11) \(h^c(x, c)\). Then we can also find a flow on a center manifold from the first equation from (8.11):

\[
\frac{dx}{dt} = F(x, h^c(x, c), c) = \text{RHS}
\] (8.13)

However now the equation which describes the flow along the center manifold will be an ODE with a parameter. And because of conditions (8.12) the equilibrium of equation (8.13) at \(x = 0, c = 0\) will be non-hyperbolic. To see it let us compute the partial derivative of the right hand side with respect to \(x\):

\[
\frac{\partial \text{RHS}}{\partial x} = \frac{\partial F}{\partial x}(0, 0, 0) + \frac{\partial F}{\partial y}(0, 0, 0) + \frac{\partial G}{\partial x}(0, 0, 0) = 0
\]

We get a zero here because each of the three terms here is zero because of conditions (8.12) and as we have assumed \(\frac{\partial F}{\partial x}(0, 0, 0) = 0\).

Now, because ODE (8.13) is a 1D ODE with a parameter which has a non-hyperbolic equilibrium at \(x = 0, c = 0\) we know that it will results in one of three possible bifurcations: fold, transcritical or pitch-fork bifurcation. We can easily study these bifurcations in the same way as we did earlier, and draw bifurcation diagrams and 1D phase portraits.

Finally we will add a flow along the \(y\) axis, which is converging at \(\lambda_2 < 0\), or diverging at \(\lambda_2 > 0\).

### 8.3.1 Center manifold for system with parameter

To prove that we can find a central manifold for a system with a parameter we will use the following mathematical trick. Let us extend our two variable system (8.11) to a three variable system:

\[
\begin{align*}
\frac{dx}{dt} &= F(x, y, c) \\
\frac{dy}{dt} &= \lambda_2(c)y + G(x, y, c) \\
\frac{dc}{dt} &= 0
\end{align*}
\] (8.14)

This system is exactly the same as the original system (8.11) as \(dc/dt = 0\) means that \(c = \text{constant}\). However \(c\) is now formally a variable and we can now find a center manifold for this system in a three dimensional space \(x, y, c\). The center manifold now will be two dimensional. So see it let us consider a linear system (8.2). In this system the center manifold \(E^c\) is the \(x\) axis. therefore in a linearization of system (8.14) which at \(c = 0\) is

\[
\begin{align*}
\frac{dx}{dt} &= 0 \\
\frac{dy}{dt} &= \lambda_2(0)y \\
\frac{dc}{dt} &= 0
\end{align*}
\]
the $E^c$ will now be a 2D plane $(x, c)$, (fig.3a). Therefore the center manifold in a non-linear system (8.14) $W^c$ will be a surface in a 3D space, tangent to the $E^c$, fig.3b. The equation of such surface is:

$$y = h^c(x, c)$$

![Figure 8.3: The center manifolds in a linear (a) and in a nonlinear (b) system with parameter.](image)

This surface has a special structure. If our trajectory starts at this surface, it will move along this surface parallel to the $x$ axis. This is because $dc/dt = 0$ and for our trajectory $c = constant$. Therefore the invariant manifold $W^c$ consists of lines parallel to the $x$ axis, or lines $y = h^c(x, c)$ where $c$ is a constant. So, we found invariant manifolds for a system with a parameter for each value of $c$.

### 8.4 Fold bifurcation in a two-variable system

It turns out, that we can find general conditions of the fold bifurcation in two-variable systems. Let us study the system which we assume is transformed into a canonical form:

$$\begin{align*}
\frac{dx}{dt} & = F(x, y, c) \\
\frac{dy}{dt} & = \lambda_2(c)y + G(x, y, c)
\end{align*}$$  \hspace{1cm} (8.15)

and $\frac{\partial F}{\partial x}(0,0,0) = \frac{\partial F}{\partial y}(0,0,0) = 0$.

Then, as we know, we need to find a center manifold $y = h^c(x, c)$ and study the flow on this center manifold.

Assume that the center manifold is $y = h^c(x, c)$. It will satisfy the following condition:

$$\frac{\partial h^c(x, c)}{\partial x}(0,0) = 0$$  \hspace{1cm} (8.16)

This condition is valid because at $c = 0$ we have a center manifold in a system without parameter (8.3) and this manifold is tangent to the $x$ axis because of the Theorem 8.

It turns out, that this condition is the only one which we need to study the fold bifurcation in system (8.15).

The flow on the center manifold is given by:

$$\frac{dx}{dt} = F(x, h^c(x, c), c) = RHS(x, c)$$  \hspace{1cm} (8.17)
This is a one variable equation with a parameter. As we have discussed in section 8.3 this equation has a non-hyperbolic equilibrium at the point \(x = 0, c = 0\), i.e., \(RHS(0, 0) = 0, \frac{\partial RHS}{\partial x}(0, 0) = 0\). (See text below eq.8.13). In order to claim the fold bifurcation here we need to check the non-degeneracy conditions for this bifurcation. These conditions are:

\[
\frac{\partial RHS}{\partial c}(0, 0) \neq 0 \quad \frac{\partial^2 RHS}{\partial x^2}(0, 0) \neq 0 \tag{8.18}
\]

For the first condition we get:

\[
\frac{\partial RHS}{\partial c}(0, 0) = \frac{\partial F}{\partial x} \frac{\partial x}{\partial c} + \frac{\partial F}{\partial y} \frac{\partial h^c(x, c)}{\partial c} + \frac{\partial F}{\partial c}(0, 0) = \frac{\partial F}{\partial c} \tag{8.19}
\]

this is zero because \(\frac{\partial F}{\partial x} = 0\), the next term is zero as \(\frac{\partial F}{\partial y} = 0\) from (8.12).

Similarly

\[
\frac{\partial^2 RHS}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial h^c(x, c)}{\partial x} \right) = \frac{\partial^2 F}{\partial x^2} + \frac{\partial F}{\partial y} \frac{\partial^2 h^c(x, c)}{\partial x^2} + \frac{\partial^2 F}{\partial h^c(x, c)} \frac{\partial^2 h^c(x, c)}{\partial x^2} = \frac{\partial^2 F}{\partial x^2}(0, 0) \tag{8.20}
\]

here the dropped terms are zeros because of conditions (8.13) and condition (8.16).

Now, we can apply the theorem of fold bifurcation and state that:

\[
\frac{\partial F}{\partial c}(0, 0) \neq 0 \quad \frac{\partial^2 F}{\partial x^2}(0, 0) \neq 0 \quad \text{if}
\]

then close to \((0, 0)\) equation (8.15) is locally equivalent to one of the following normal forms:

\[
d\eta/dt = \pm \mu \pm \eta^2 \tag{8.22}
\]

and the tangent bifurcation takes place.

The sign at \(\mu\) is the same as the sign of \(\frac{\partial F}{\partial c}\), and the sign at \(\eta^2\) is the same as the sign of \(\frac{\partial^2 F}{\partial x^2}\).

### 8.4.1 Theorem. Tangent (Saddle-node, Fold) bifurcation for two variable ODEs

Let the system

\[
\begin{aligned}
&dx/dt = f(x, y, c) \\
&dy/dt = g(x, y, c)
\end{aligned} \tag{8.23}
\]

has an equilibrium point \(x = x^*, y = y^*, c = c^*\), with

\[
\lambda_1(c^*) = 0 \quad \lambda_2(c^*) \neq 0 \tag{8.24}
\]

Assume that in canonical form system (8.23) is:

\[
\begin{aligned}
&dx/dt = F(x, y, c) \\
&dy/dt = \lambda_2(c)y + G(x, y, c)
\end{aligned} \tag{8.25}
\]

and \(\frac{\partial F}{\partial x}(0, 0, 0) = \frac{\partial F}{\partial y}(0, 0, 0) = 0\).

\[
\frac{\partial F}{\partial c}(x^*, c^*) \neq 0 \quad \frac{\partial^2 F}{\partial x^2}(x^*, c^*) \neq 0 \tag{8.26}
\]
then close to \((x^*, y^*, c^*)\) the flow of the system on a center manifold is locally equivalent to one of the following normal forms:

\[
d\eta/dt = \pm \mu \pm \eta^2
\]  

(8.27)

and the tangent bifurcation takes place.

Note: the sign at \(\mu\) is the same as the sign of \(\partial F / \partial c\), and the sign at \(\eta^2\) is the same as the sign of \(\partial^2 F / \partial c^2\).

Bifurcation diagrams are the same as in 1D case. The phase portraits for system (8.25) can be drawn by combining 1D flow on center manifold with hyperbolic flow along the \(y\) axis.

### 8.4.2 Practical notes

We have found the conditions for the fold bifurcation in a two variable system. We see, that the fold bifurcation has necessary and non-degeneracy conditions.

The main necessary condition is \(\lambda_1(c^*) = 0\). It can be easily evaluated form the following fact.

\[
det J = \lambda_1 \ast \lambda_2
\]  

(8.28)

where \(J\) is the Jacobian of our system at the equilibrium.

Because \(\lambda_1 = 0\), the necessary condition for the fold bifurcation is

\[
det J = 0
\]  

(8.29)

Unfortunately, is it not so easy to check the non-degeneracy conditions. This is because to apply formula (8.26) one needs first to transform the non-linear system (8.23) to its canonical form (8.25).

### 8.5 Other bifurcations

Other bifurcation which can occur when \(\lambda_1 = 0\) can also be studied in a similar way. For that we need to find the center manifold, find the flow of our system along this center manifold and study bifurcation in the obtained one variable system. Unfortunately, we cannot get simple general expressions for non-degeneracy conditions for other bifurcation similar to conditions (8.21) for the fold bifurcation. Therefore in order to study the transcritical, or pitchfork bifurcation analytically for a particular system, you do need to find the center manifold and study the flow on that center manifold for your particular system.
Chapter 9

1D maps. Main definitions.

9.1 1D maps without parameter.

Let us start consideration of the second class of dynamical systems, which are called maps, or discrete time dynamical systems. Such maps naturally occur in many real situations. Assume that we initially have \( N_0 \) bacteria. Then if each bacterium divides into two, the next generation of bacteria \( N_1 \) will be:

\[
N_1 = 2N_0
\]

\[
then
\]

\[
N_2 = 2N_1
\]

\[
\ldots
\]

\[
N_{n+1} = 2N_n
\]

\[
\ldots
\]

Let us write it in a general way. If the next generation of a population is a function \( f \) from the previous generation, we can represent the dynamics of the population \( x \) as:

\[
x_{n+1} = f(x_n)
\]

Here \( n \) is the generation of our population which can be viewed as a discrete time variable. In order to stress that \( n \) is similar to time we will use the letter \( t \) for the generation number, however here \( t \) takes only integer values 1, 2, 3, ... etc.:

\[
x_{t+1} = f(x_t)
\]

(9.1)

The other representation of the map is the following.

\[
x_1 = f(x_0)
\]

\[
x_2 = f(x_1) = f(f(x_0)) = f^{(2)}(x_0)
\]

\[
x_3 = f(x_2) = f(f^{(2)}(x_0)) = f^{(3)}(x_0)
\]

\[
\ldots
\]

\[
x_t = f^{(t)}(x_0)
\]

(9.2)

Note, that \( f^{(2)}(x_0) \), and \( f^{(t)}(x_0) \) are not derivatives, or powers of our function, they are just 2nd and \( t \)-th iterates. For example if \( f(x) = \sin x \), then \( f^{(2)}(x) = \sin(\sin x) \).

There are two main ways of graphical representation of dynamics of map (9.1). One is similar to the phase portrait of 1D ODEs. We use the x-axis, put there the successive values of the variable \( x \) and connect them. For example, if we consider the map describing bacteria
growth $x_{t+1} = 2x_t$ and if we want to represent dynamics of population with the initial size $x_0 = 3$, we will get $x_1 = 6; x_2 = 12; x_3 = 24$ etc. The graphical representation of this dynamics is in fig.9.1.

\[ \begin{array}{c}
0 & 3 & 6 & 12 & 24 & x \\
\hline
x_0 & x_1 & x_2 & x_3 & \\
\end{array} \]

Figure 9.1: The dynamics of the map $x_{t+1} = 2x_t$ with $x_0 = 3$

However in the most of the cases we use the other type of representation which is called a “cobweb” representation, probably because it looks like a network spread by a spider around the graph of the function $f(x)$. In this representation we draw the graph of the function $y = f(x)$, and the line $y = x$. (fig.9.2) Then we start with our initial condition (point $x = 3$) and go vertically to the graph of our right hand side function $f(x)$. The point of intersection has the $y$ coordinate $y = f(x) = 2x = 6$ which is $x_1$ (the next iteration of our map.) However, in order to have $x_1$ on the $x$-axis we go horizontally till intersection with the line $y = x$. As on this line $y = x$ then the $x$ coordinate of the point of intersection will be $x_1 = y_1 = f(x_0)$, or we get as we wanted the next generation of our population on the $x$-axis. If we continue this process further: vertical until $y = f(x)$, then horizontal until $y = x$ we will get successive points $x_2, x_3, \cdots, x_t$, etc. So we represented a dynamics of our map.

\[ \begin{array}{c}
0 & 3 & 6 & 12 & 24 & x \\
\hline
x_0 & x_1 & x_2 & x_3 & \\
\end{array} \]

Figure 9.2: The cobweb graph for the map $x_{t+1} = 2x_t$ with $x_0 = 3$

In many cases the cobweb representation is more informative than the usual representation from fig.9.1. This is because even simple 1D maps can have a complex dynamics. Consider, for example the map $x_{t+1} = -2x_t$. Its cobweb representation is given in fig.9.3.

We see that in this case we have oscillations of $x$ around the point $x = 0$ and its cobweb representation looks like a “squared spiral”. So, in some way, we can have a “spiral” even in 1D maps. Note, that for ODEs spiral occurs in 2D only.

Let us continue study of 1D maps in a way similar to our study of ODEs. For 1D ODEs the most important points on the phase portrait were the equilibria points. The rate of change of variables at the equilibrium point was zero, and the dynamics was extremely simple: $x(t) = x_{eq}$. Similar points exist also for 1D maps. In fact, to have such dynamics we need to fulfill just one simple requirement: the next iteration of our map is the same as previous, or $x_{t+1} = x_t$, but because $x_{t+1} = f(x_t)$ we get the following definition for the equilibrium (for the maps it is called a fixed point):

**Definition 14** A point $x^*$ is called a fixed point of the map $x_{t+1} = f(x_t)$, if $f(x^*) = x^*$
Figure 9.3: The cobweb graph for the map $x_{t+1} = -2x_t$.

What does it mean geometrically? We can rewrite the condition $f(x^*) = x^*$ as two conditions:

$$\begin{cases} y = f(x) \\ y = x \end{cases}$$

(9.3)

So, the fixed point is a intersection of the graph of our function $y = f(x)$ with the line $y = x$. Fig.9.4 shows various examples of graphical identification of the fixed points.

Figure 9.4: The graphical finding of fixed points of the map $x_{t+1} = f(x_t)$. (a) two fixed points; (b) no fixed points; (c) 5 fixed points;

The next important step in study of ODEs was finding equilibrium type. In 1D ODEs we had just two main types of equilibria: stable and non-stable equilibria. Situation in the case of maps is more complex, however we can still use notion of a stable and a non-stable fixed point. Let us consider a simple example of a linear map:

$$x_{t+1} = ax_t$$

The fixed point of this map is given by:

$$\begin{align*} x^* &= ax^* \\ x^* &= 0 \text{ if } a \neq 0 \\ x^* &= \text{any number if } a = 1 \end{align*}$$

(9.4)
The dynamics of our map is very simple:

\[ \begin{align*}
  x_1 &= ax_0 \\
  x_2 &= ax_1 = a^2x_0 \\
  x_3 &= ax_2 = a^3x_0 \\
  &\vdots \\
  x_t &= a^tx_0
\end{align*} \]

Because \( x_0 \) is a constant, the dynamics type depends on behavior of the exponential function \( a^t \). We know that if \( a > 0 \) this function approaches 0 if \( 0 \geq a < 1 \), goes to infinity if \( a > 1 \), or is constant if \( a = 1 \). The graph of this function is given in Fig. 9.5. For \( 0 \geq a < 1 \), \( x_t \) will approach 0 (which is the only fixed point at \( a \neq 1 \)) and we have a stable equilibrium. If \( a > 1 \), \( x_t \) will diverge to infinity and equilibrium is unstable. For \( a = 1 \) any number \( x \) is an equilibrium, so each trajectory which starts at \( x_0 \) is a trajectory which starts at equilibrium point, i.e., \( x_t \equiv x_0 \).

![Graph of the function \( a^t \).](attachment:fig9.5.png)

Now, let us consider the case of negative \( a < 0 \). Dynamics here is similar to the dynamics for the case \( a > 0 \), i.e., \(-1 < a < 0\) gives us stable equilibrium and \( a < -1 \) corresponds to unstable equilibrium. The only difference from the case of \( a > 0 \) occurs because for \( a < 0 \), \( a^t = (-|a|^t)^t = (-1)^t|a|^t \), and we have oscillations around the point \( x = 0 \). The cobweb graphs of such behaviors are shown in Fig. 9.6. We see that the case \(-1 < a < 0\) looks like stable “squared spiral” and the case \( a < -1 \) looks like unstable “squared spiral”. In the case \( a = -1 \) the dynamics is \( x_t = (-1)^tx_0 \), which means: \( x_1 = -x_0; x_2 = +x_0; x_3 = -x_0; x_4 = x_0; \text{etc.} \), so we have periodic trajectory for each \( x_0 \) (see Fig. 9.6). These trajectories look like “squared center point” for 2D ODEs. As in the case of 2D ODEs the “squared center point” separates the cases of stable and non-stable spirals. As in the case of 2D ODEs, such center point has a neutral stability, i.e., in some sense it is both stable and non-stable.

We can conclude the results of our analysis for both cases \( a > 0 \) and \( a < 0 \) in the following theorem:

**Theorem 9** For a linear 1D map \( x_{t+1} = ax_t \), the fixed point \( x_0 = 0 \) is stable if \( |a| < 1 \) and unstable if \( |a| > 1 \).

Now let us consider a general nonlinear map (9.1): \( x_{t+1} = f(x_t) \). We can always shift a fixed point of this map to the point \( x = 0 \). Therefore we can assume that for our map \( 0 = f(0) \). Let us expand the function \( f(x) \) close to zero into the Taylor series up to the first order:

\[ f(x) \approx f(0) + \frac{df}{dx}(0) * x + \cdots \]
Because $0 = f(0)$ the Taylor expansion becomes: $f(x) \approx \frac{df}{dx}(0) \cdot x$, so our map is similar to a linear map (9.4). Therefore the theorem on stability of nonlinear is the following:

**Theorem 10** Suppose that $x^*$ is a fixed point of $x_{t+1} = f(x_t)$, then the fixed point $x^*$ is stable if $|f'(x^*)| < 1$, is non-stable if $|f'(x^*)| > 1$ and no information if $|f'(x^*)| = 1$

Note, that as in the case of linear map, the case $|f'(x^*)| = 1$ is special, as it gives a very special phase portrait which can be disturbed by small perturbations. Therefore it is not unexpected for us, that this case give a non-hyperbolic situation:

**Definition 15** A fixed point $x^*$ of $x_{t+1} = f(x_t)$ is called hyperbolic if $| \frac{df}{dx}(x^*) | \neq 1$

And as for usual hyperbolic case we have the following theorem:

**Theorem 11** Let $x^*$ be a hyperbolic fixed point of the map $x_{t+1} = f(x_t)$. Then there is a neighborhood of this fixed point, where this nonlinear map is equivalent (topologically conjugate) to the linearized map.

### 9.2 1D map with a parameter

Let us consider a map with a parameter:

$$x_{t+1} = f(x_t, c) \quad (9.5)$$

We can study this map at some fixed parameter value, find fixed points and study their hyperbolicity.

Here it is also possible to prove the following theorem

**Theorem 12** Let $x^*, c^*$ be a hyperbolic fixed point of the map $x_{t+1} = f(x_t, c)$. Then at some close values of $c$ we will also have a hyperbolic fixed point with the same stability.

Therefore hyperbolic points are not interesting for us. Consider non-hyperbolic fixed points. There are the following non-hyperbolic cases:

$$\frac{\partial f}{\partial x}(x^*, c^*) = 1 \quad (9.6)$$
and
\[ \frac{\partial f}{\partial x}(x^*, c^*) = -1 \]  \hspace{1cm} (9.7)

Case (9.6) will give us the fold, transcritical and the pitch-fork bifurcation. Case (9.7) will give us the flip bifurcation.

Let us first study case (9.6).
Chapter 10

Fold bifurcation for maps

10.1 Normal form

Consider the map:

\[ x_{t+1} = f(x_t, c) \]  \hfill (10.1)

Assume that \( f(x, c) \) has a non-hyperbolic fixed point at \( x = x^*, c = c^* \):

\[ f(x^*, c^*) = x^* \quad \frac{\partial f}{\partial x}(x^*, c^*) = 1 \]

We know, that we can always shift this equilibrium point to the point \( x = 0, c = 0 \). In that case the conditions for such non-hyperbolicity will be:

\[ f(0, 0) = 0 \quad \frac{\partial f}{\partial x}(0, 0) = 1 \]  \hfill (10.2)

Note, that the first condition \( f(0, 0) = 0 \) looks similar to conditions for equilibria for ODEs. However, you should remember, that this is just coincidence, as the general condition for a fixed point of a map is \( f(x^*, c^*) = x^* \), and it coincides with the condition for an equilibrium point of an ODE \( f(x^*, c^*) = 0 \), only if \( x^* = 0 \).

The study of this case is very similar to the study of fold bifurcation of ODEs from chapter3.

We need to expand the right hand side of map (10.1) into the Taylor series and simplify the obtained expression (find the normal form).

\[ f(x, c) = f(0, 0) + \frac{\partial f}{\partial x}(0, 0)x + \frac{\partial f}{\partial c}(0, 0)c + + \frac{\partial^2 f}{\partial x^2} \frac{x^2}{2} + \frac{\partial^2 f}{\partial x \partial c} xc + \frac{\partial^2 f}{\partial c^2} \frac{c^2}{2} + O((x, c)^3) \]  \hfill (10.3)

Because of conditions (10.2) the expansion becomes:

\[ f(x, c) = x + \frac{\partial f}{\partial c} c + \frac{\partial^2 f}{\partial x^2} \frac{x^2}{2} + \frac{\partial^2 f}{\partial x \partial c} xc + \frac{\partial^2 f}{\partial c^2} \frac{c^2}{2} + O((x, c)^3) \]  \hfill (10.4)

Or the map becomes:

\[ x_{t+1} = x_t + \frac{\partial f}{\partial c} c + \frac{\partial^2 f}{\partial x^2} \frac{x_t^2}{2} + \frac{\partial^2 f}{\partial x \partial c} x_t c + \frac{\partial^2 f}{\partial c^2} \frac{c^2}{2} + O((x, c)^3) \]  \hfill (10.5)

It can be proved that we can omit the higher order terms which are collected in \( O((x, c)^3) \).
Now we make normalization of the map. As for the fold bifurcation for ODEs we will remove the term \(\frac{\partial^2 f}{\partial c \partial x} x_c\) which is linear with respect to the variable \(x\). For that we introduce a new variable \(\xi\)

\[
\xi = x - \delta
\]  

(10.6)

where \(\delta\) is unknown parameter which we will fix in order to make normalization. We will also need the inverse transformation:

\[
x = \xi + \delta
\]  

(10.7)

Our plan for transformations is the same as for ODEs:

1. Use the direct transformation \(\xi(x, \delta)\) and find \(\xi_{t+1}\). We will find an expression which will include \(x_t, x_{t+1}\).

2. Express \(x_{t+1}\) using map (10.5). We will get a map of the form \(\xi_{t+1} = F(x_t, \delta, c)\)

3. Replace \(x\) in the right hand side by the inverse transformation \(x(\xi, \delta)\). After that we will get the map in the new coordinates.

4. Fix the unknown parameter \(\delta\) to simplify the equation.

Let us do it

1. Use the direct transformation (10.6). Because \(\delta\) is a parameter (constant) we get:

\[
\xi_{t+1} = x_{t+1} - \delta
\]

2. Express \(x_{t+1}\) using map (10.5).

\[
\xi_{t+1} = x_{t+1} - \delta = x_t - \delta + \frac{\partial f}{\partial c} c + \frac{\partial^2 f}{\partial x^2} \frac{x_t^2}{2} + \frac{\partial^2 f}{\partial x \partial c} x_c + \frac{\partial^2 f}{\partial c^2} \frac{c^2}{2}
\]

3. Replace \(x\) in the right hand side by the inverse transformation (10.7)

As \(x = \xi + \delta\) we get

\[
\xi_{t+1} = \xi_t + \delta - \delta + \frac{\partial f}{\partial c} c + \frac{\partial^2 f}{\partial x^2} \left(\frac{(\xi_t + \delta)^2}{2}\right) + \frac{\partial^2 f}{\partial x \partial c} (\xi_t + \delta)c + \frac{\partial^2 f}{\partial c^2} \frac{c^2}{2}
\]

\[
= \xi_t + \left(\frac{\partial f}{\partial c} c + \frac{\partial^2 f}{\partial x^2} \frac{\xi_t^2}{2} + \frac{\partial^2 f}{\partial x \partial c} \delta c + \frac{\partial^2 f}{\partial c^2} \frac{\delta^2}{2}\right)
\]

\[
+ \xi_t \left(\frac{\partial^2 f}{\partial x \partial c} c + \frac{\partial^2 f}{\partial x^2} \delta\right)
\]

\[
+ \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \xi_t^2
\]

4. Fix the unknown parameter \(\delta\) to simplify the equation.

Because we need to remove linear terms \(\xi (\frac{\partial f}{\partial x \partial c} c + \frac{\partial^2 f}{\partial x^2} \delta)\) we require:

\[
\frac{\partial^2 f}{\partial x \partial c} c + \frac{\partial^2 f}{\partial x^2} \delta = 0
\]
\[ \delta = -\frac{\partial^2 f}{\partial x \partial c} \]  

(10.8)

We can do it if:

\[ \frac{\partial^2 f}{\partial x^2} \neq 0 \]  

(10.9)

After that our map becomes:

\[ \xi_{t+1} = \xi_t + T_0 + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \xi_t^2 \]  

(10.10)

where

\[ T_0 = \left( \frac{\partial f}{\partial c} c + \frac{\partial^2 f}{\partial x^2} \frac{c^2}{2} + \frac{\partial^2 f}{\partial x \partial c} \delta c + \frac{\partial^2 f}{\partial x^2} \frac{\delta^2 c}{2} \right) \]

If we substitute here the value of \( \delta \) from (10.8) and simplify the expression we get:

\[ T_0 = \frac{\partial f}{\partial c} c + c^2 \left( \frac{1}{2} \frac{\partial^2 f}{\partial x^2} - \left( \frac{\partial^2 f}{\partial x \partial c} \right)^2 - \frac{\partial^2 f}{\partial x^2} \right) \]  

(10.11)

We will further simplify this map by introduction a new parameter and by “rescaling of amplitude”

### 10.1.1 New parameter

As for the fold bifurcation for ODE we introduce

\[ \beta = T_0 \quad \text{if} \quad \frac{\partial f}{\partial c} > 0 \quad \beta = -T_0 \quad \text{if} \quad \frac{\partial f}{\partial c} < 0 \]  

(10.12)

Now our map becomes:

\[ \xi_{t+1} = \xi_t + \beta + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \xi_t^2 \]  

(10.13)

### 10.1.2 Rescaling of amplitude

If we now introduce a new parameter

\[ \eta = \left| \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right| \xi \]  

(10.14)

our map becomes:

\[ \eta_{t+1} = \left| \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right| \xi_{t+1} = \left| \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right| \xi_t + \beta \left| \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right| \xi_t^2 + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \xi_t^2 \]  

(10.15)

After the inverse substitution \( \xi = \frac{\eta}{\left| \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right|} \) we get

\[ \eta_{t+1} = \eta_t \pm \beta \left| \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right| \eta_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \left| \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right| \eta_t^2 \]  

(10.16)

If we denote \( \beta \left| \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right| \) as a new parameter \( \mu \) we get the following normal form

\[ \eta_{t+1} = \eta_t \pm \mu \pm \eta_t^2 \]  

(10.17)
10.1.3 Conclusion

We see that any map which has a non-hyperbolic fixed point can be transformed to the more simple form (10.17). However, in order to make these transformation we need that the non-degeneracy conditions be satisfied.

10.2 Study of the normal form

Let us consider the case

\[ \eta_{t+1} = \eta_t + \mu - \eta_t^2 \]

1. Fixed points.

\[ \eta + \mu - \eta^2 = \eta \]

or

\[ \eta = \pm \sqrt{\mu} \quad \text{if} \quad \mu > 0 \]

2. Stability

\[ \frac{\partial f}{\partial \eta} = 1 - 2\eta \]

\[ \frac{\partial f}{\partial \eta}(\sqrt{\mu}) = 1 - 2\sqrt{\mu} < 1 \]

i.e., fixed point \( \eta = \sqrt{\mu} \) is stable

\[ \frac{\partial f}{\partial \eta}(-\sqrt{\mu}) = 1 + 2\sqrt{\mu} > 1 \]

i.e., fixed point \( \eta = -\sqrt{\mu} \) is unstable
10.3 Theorem. Tangent (Saddle-node, Fold) bifurcation for maps

Let \( x_{t+1} = f(x_t, c) \), has a fixed point \( x = 0, c = 0 \), with

\[
\frac{\partial f}{\partial x}(0,0) = 1
\]

and

\[
\frac{\partial f}{\partial c}(0,0) \neq 0 \quad \frac{\partial^2 f}{\partial x^2}(0,0) \neq 0
\]

then close to \((0,0)\) the map is locally equivalent to one of the following normal forms:

\[
\eta_{t+1} = \eta_t \pm \mu \pm \eta_{t+1}^2
\]

and the tangent bifurcation takes place.

Note: the sign at \( \mu \) is the same as the sign of \( \frac{\partial f}{\partial c} \), and the sign at \( \eta^2 \) is the same as the sign of \( \frac{\partial^2 f}{\partial x^2} \).

Bifurcation diagrams

![Bifurcation diagrams for the fold bifurcation](image)

Figure 10.1: Bifurcation diagrams for the fold bifurcation
Chapter 11

Transcritical bifurcation

Let us consider a map:

\[ x_{t+1} = f(x_t, c) = x_t g(x_t, c) \]

As for ODEs the non-degeneracy condition

\[ \frac{\partial f}{\partial c}(0,0) \neq 0 \]

does not hold here. In this case we also get a transcritical bifurcation.

The general condition for non-hyperbolicity here is:

\[ \frac{\partial f}{\partial x} = \frac{\partial x g(x, c)}{\partial x} = g(x, c) + x \frac{\partial g(x, c)}{\partial x} = 1 \]

So, at the fixed point \( x = 0, c = 0 \) we get:

\[ g(0,0) = 1 \]

Let us study what happens at non-hyperbolic fixed point of this map.

11.1 Normal form

Consider the map:

\[ x_{t+1} = f(x_t, c) = x_t g(x_t, c) \quad (11.1) \]

Assume that \( x g(x, c) \) has a non-hyperbolic fixed point at \( x = 0, c = 0 \):

\[ g(0,0) = 1 \quad (11.2) \]

We use Taylor expansion of the right hand side of map \( (11.1) \) up to the first order:

\[ f(x, c) = x g(x, c) = x(g(0,0) + \frac{\partial g}{\partial x}(0,0)x + \frac{\partial g}{\partial c}(0,0)c) \]

\[ = x(1 + \frac{\partial c}{\partial x}(0,0)x + \frac{\partial c}{\partial c}(0,0)c) \quad as \quad g(0,0) = 1 \quad (11.3) \]

It is obvious, that if we introduce a new parameter and make rescaling of the amplitude, in the same way as we did for the analysis of the fold bifurcation, we get the following normal form:

\[ \eta_{t+1} = \eta_t + \eta_t(\pm \mu \pm \eta_t) \quad (11.4) \]

The obvious non-degeneracy conditions for normalization are:

\[ \frac{\partial g}{\partial x}(0,0) \neq 0 \quad \frac{\partial g}{\partial c}(0,0) \neq 0 \quad (11.5) \]
11.2 Study of the normal form

Let us consider the case

\[ \eta_{t+1} = \eta_t + \eta_t (\mu - \eta_t) \]

1. Equilibria.

\[ \eta + \eta (\mu - \eta) = \eta \]

or

\[ \eta = 0 \quad \eta = \mu \]

2. Stability

\[ \frac{\partial f}{\partial \eta} = 1 + \mu - 2\eta \]

\[ \frac{\partial f}{\partial \eta}(0) = 1 + \mu \]

i.e., fixed point \( \eta = 0 \) is stable for \( \mu < 0 \) and unstable for \( \mu > 0 \)

The second fixed point \( \eta = \mu \):

\[ \frac{\partial f}{\partial \eta}(\mu) = 1 - \mu \]

i.e., fixed point \( \eta = \mu \) is stable for \( \mu > 0 \) and unstable for \( \mu < 0 \)

The graphs of this map are shown in fig.11.1.

![Figure 11.1: Transcritical bifurcation for the map \( \eta_{t+1} = \eta_t + \eta_t (\mu - \eta_t) \). Left \( \mu = -0.5 \), right \( \mu = 0.5 \).](image)

11.3 Theorem. Transcritical bifurcation for maps

Let \( x_{t+1} = f(x_t, c) = x_t g(x_t, c) \), has a non-hyperbolic fixed point \( x = 0, c = 0 \):

\[ g(0, 0) = 1 \]

and

\[ \frac{\partial g}{\partial c}(0, 0) \neq 0 \quad \frac{\partial g}{\partial x}(0, 0) \neq 0 \]
then close to \((0,0)\) the map is locally equivalent to one of the following normal forms:

\[
\eta_{t+1} = \eta_{t} \pm \mu \eta_{t} \pm \eta_{t}^2
\]

and the transcritical bifurcation takes place.

Note: the sign at \(\mu\) is the same as the sign of \(\frac{\partial g}{\partial c}\), and the sign at \(\eta^2\) is the same as the sign of \(\frac{\partial g}{\partial x}\).

Figure 11.2: Bifurcation diagrams for the transcritical bifurcation
Chapter 12

Pitchfork bifurcation for maps

12.1 Normal form

Consider the map:

$$x_{t+1} = f(x_t, c)$$  \hspace{1cm} (12.1)

Assume that $f(x, c)$ is an odd function:

$$f(-x, c) = -f(x, c)$$  \hspace{1cm} (12.2)

Then, as we know from the analysis of the pitchfork bifurcation of ODEs performed in chapter 5, $f(0, c) = 0$ because it is an odd function (see (5.1)). Therefore, we can conclude that map (12.1) always has a fixed at $x = 0$. If this fixed point is hyperbolic, then the situation is not interesting, and small changes of the parameter do not change the dynamics of the system around this fixed point. However, if the fixed point is not hyperbolic, we need to study bifurcation. So let us assume non-hyperbolicity of the fixed point at $x = 0$:

$$f(0, c) = 0, \quad \frac{\partial f}{\partial x}(0, 0) = 1.$$  \hspace{1cm} (12.3)

As it was for ODEs the fold bifurcation is not possible here because the both non-degeneracy conditions for the fold bifurcation are not satisfied (see (5.10)). Therefore we need to find a normal form for map (12.1). Note, that because $f$ is an odd function, all terms in Taylor series which involve $x$ to the even powers (i.e., 0, 2, 4, etc.) equal to zero (see (5.8)). Taylor expansion of the right hand side is:

$$f(x, c) = f(0, 0) + \frac{\partial f}{\partial x}(0, 0)x + \frac{\partial f}{\partial c}(0, 0)c + \frac{\partial^2 f}{\partial x^2} x^2 + \frac{\partial^2 f}{\partial x \partial c} xc + \frac{\partial^2 f}{\partial c^2} c^2 + \frac{\partial^3 f}{\partial x^3} x^3 + \frac{\partial^3 f}{\partial x^2 \partial c} x^2 c + \frac{\partial^3 f}{\partial x \partial c^2} xc^2 + \frac{\partial^3 f}{\partial c^3} c^3 \cdots$$  \hspace{1cm} (12.4)

Because our function is odd the derivatives as even powers of $x$ are zeros (5.8):

$$f(0, 0) = 0, \quad \frac{\partial f}{\partial c} = 0, \quad \frac{\partial^2 f}{\partial x^2} = 0, \quad \frac{\partial^2 f}{\partial x \partial c} = 0, \quad \frac{\partial^2 f}{\partial c^2} = 0, \quad \frac{\partial^3 f}{\partial x^3} = 0,$$  \hspace{1cm} (12.5)
Because equilibrium is non-hyperbolic $\frac{\partial f}{\partial x} = 1$.
Substituting these into (12.4) yields:

$$x_{t+1} = x_t + x_t\left(\frac{\partial^2 f}{\partial x \partial c} c + \frac{\partial^3 f}{\partial x \partial c^2} \frac{c^2}{2}\right) + \frac{\partial^3 f}{\partial x^3} \frac{x_t^3}{6}$$  \hspace{1cm} (12.6)

Now, let us introduce a new parameter

$$\beta = \frac{\partial^2 f}{\partial x \partial c} c + \frac{\partial^3 f}{\partial x \partial c^2} \frac{c^2}{2} \text{ or } \frac{\partial^2 f}{\partial x \partial c} > 0$$

or

$$-\beta = \frac{\partial^2 f}{\partial x \partial c} c + \frac{\partial^3 f}{\partial x \partial c^2} \frac{c^2}{2} \text{ or } \frac{\partial^2 f}{\partial x \partial c} < 0$$

and after rescaling the amplitude we get the following normal form:

$$\eta_{t+1} = \eta_t \pm \mu \eta_t \pm \eta_t^3$$  \hspace{1cm} (12.7)

### 12.2 Study of the normal form

Let us consider the case

$$\eta_{t+1} = \eta_t + \mu \eta_t - \eta_t^3$$

1. Fixed points

$$\eta + \mu \eta - \eta^3 = \eta$$

or

$$\eta = 0; \quad \eta = \pm \sqrt{\mu} \quad i f \quad \mu > 0$$

2. Stability

$$\frac{\partial f}{\partial \eta} = 1 + \mu - 3\eta^2$$

$$\frac{\partial f}{\partial \eta}(0) = 1 + \mu$$

i.e., fixed point $\eta = 0$ is stable at $\mu < 0$ and is unstable at $\mu > 0$

$$\frac{\partial f}{\partial \eta} = 1 + \mu - 3\eta^2$$

$$\frac{\partial f}{\partial \eta}(\pm \sqrt{\mu}) = 1 + \mu - 3\mu = 1 - 2\mu < 0 \text{ for } \mu > 0$$

i.e., both fixed points $\eta = \pm \sqrt{\mu}$ are stable.

3. Note, that at $\mu = 0$ the fixed point is stable. In general the stability of the fixed point is determined by the sign at $\pm \eta^3$, i.e., if the sign is negative the fixed point at $\mu = 0$ is stable, if the sign is positive the fixed point at $\mu = 0$ is unstable. The sign at $\pm \eta^3$ is determined by the sign of $\frac{\partial^3 f}{\partial x^3}$ (see (12.6)).

The graphs of this map are shown in fig.12.1.
Figure 12.1: Pitchfork bifurcation for the map $x_{t+1} = x_t + \eta_t (\mu - \eta_t^2)$. Left $\mu = -0.5$, right $\mu = 0.5$.

### 12.3 Theorem. Pitchfork bifurcation for maps.

Consider the map $x_{t+1} = f(x_t, c)$, such that $f(-x, c) = -f(x, c)$ for all $c$ close to $c = 0$. If this equation has a non-hyperbolic fixed point at $x = 0, c = 0$:

$$\frac{\partial f}{\partial x}(0, 0) = 1$$

and if

$$\frac{\partial^2 f}{\partial x \partial c}(0, 0) \neq 0 \quad \frac{\partial^3 f}{\partial x^3}(0, 0) \neq 0$$

then close to $(0, 0)$ this equation is locally equivalent to one of the following normal forms:

$$x_{t+1} = x_t + \eta_t (\pm \mu \pm \eta_t^2)$$

and the pitchfork bifurcation takes place.

1. Note: the sign at $\mu$ is the same as the sign of $\frac{\partial^2 f}{\partial x \partial c}$, and the sign at $\eta^2$ is the same as the sign of $\frac{\partial^3 f}{\partial x^3}$.

2. Note, that stability of fixed point at bifurcation point determines the stability of the new born solution, i.e., if the fixed point at $\mu = 0$ is stable, than the 2 new fixed points are also stable and vise versa.

3. Note, that the cases, when we have stable non-trivial fixed points are called supercritical pitchfork bifurcation, the cases when we have unstable non-trivial fixed points are called sub-critical pitchfork bifurcation.
Figure 12.2: Bifurcation diagrams for the pitchfork bifurcation
Chapter 13

Flip bifurcation

Now let us study bifurcation around non-hyperbolic fixed point \((x^*, c^*)\) of the map:

\[ x_{t+1} = F(x_t, c) \quad (13.1) \]

where

\[ \frac{\partial F}{\partial x}(x^*, c^*) = -1 \quad (13.2) \]

Let us assume that if \(c < c^*\), \(\frac{\partial F}{\partial x}(x^*, c^*) > -1\), at the bifurcation point \(\frac{\partial F}{\partial x}(x^*, c^*) = -1\) and if \(c > c^*\), \(\frac{\partial F}{\partial x}(x^*, c^*) < -1\). This means, that at \(c < c^*\) we have a stable equilibrium, and because the slope \(\frac{\partial F}{\partial x}\) at the bifurcation point is negative, this equilibrium is a stable “squared spiral” on the cobweb graph (fig.13.1a). At \(c > c^*\) the equilibrium becomes unstable “squared spiral” (fig.13.1b).

![Figure 13.1: Dynamics of the map \(x_{t+1} = F(x_t, c)\) close to the fixed point where \(\frac{\partial F}{\partial x}(x^*, c^*) = -1\).](image)

We know, that if such consequence of events occurs for 2D ODEs, i.e., a stable spiral becomes unstable, we usually have the Hopf bifurcation, or occurrence of a limit cycle in our system. It turns out that here we have a similar situation. We will also get a formation of a limit cycle, however this limit cycle will be also a “squared limit cycle” (fig.13.1c).

How can we describe this “squared limit cycle”? It is easy to see, that this “squared limit cycle” is just a periodic orbit of our map with a period 2. In fact, if a point \(x_1\) is on this limit cycle, then the next iteration will bring the point to the point \(x_2\) and then again to the point \(x_1\), so we will return to the initial point \(x_1\) after two iterations. The time 2, which was necessary for this return is the period of our orbit. Of course we can start iterations from the other point on this limit cycle, point \(x_2\). We will get similar behavior \(x_2; x_1; x_2; x_1; x_2; \ldots\) etc. There
is a very powerful method of description of such type of dynamics. Our original map is \(x_{t+1} = F(x_t)\). Let us introduce a double iterated map given by the function \(x_{t+1} = F(F(x_t)) \equiv F^2(x_t)\). Dynamics of this map \(x_{t+1} = F^2(x_t)\) is very similar to the dynamics of the original map, but it jumps two point ahead at each iteration. If our original map had the following successive values: \(x_0;x_1;x_2;x_3;x_4;x_5;x_6;x_7;x_8;\text{etc.}..\), then the double iterated map will have the following successive points: \(x_0;x_2;x_4;x_6;x_8;\text{etc.}..\). Now, for a point on a “squared limit cycle” with the trajectory \(x_2;x_1;x_2;x_1;x_2;x_1;x_2;\text{etc.}..\), the corresponding dynamics of a double iterated map will be just \(x_2;x_2;x_2;\text{etc.},\) or \(x_2 = F^2(x_2)\). This means that the point \(x_2\) is the fixed point of the map \(x_{t+1} = F^2(x_t)\). Similarly \(x_1 = F^2(x_1)\) and \(x_1\) is also a fixed point of a double iterated map. If some map has a periodic orbit with a period 3, this means that we return to the same point after three jumps, or this orbit will be a fixed point of a triple iterated map. In general there is the following definition for the periodic orbits (points) for discrete maps:

**Definition 16** A point \(x^*\) is called a periodic point of minimal period \(n\) of the map \(x_{t+1} = F(x_t)\), if \(F^n(x^*) = x^*\).

In the next section we will perform a systematic study of dynamics of 1D map around the point where \(\frac{\partial F}{\partial x}(x^*, c^*) = -1\). We will show that around this point we indeed get bifurcations to periodic orbits. This bifurcation is called a flip bifurcation. Our first step in studying of this bifurcation is to find a normal form for map (13.1). We call it a normal form for a single iterated map. Later we will also find a normal form for a double iterated map in order to study period points of (13.1).

### 13.1 Normal form

#### 13.1.1 Equilibrium shift

The slope of the graph of \(F(x, c)\) at the fixed point \((0, 0)\) is \(-1\), (see condition (13.2)). This means that the fixed point \((0, 0)\) must exist for all \(c\) close to \(c = 0\). We can see it from a simple computation. If we assume that we have a fixed point at \(x = 0, c = 0\), i.e., \(F(0, 0) = 0\), and \(\frac{\partial F}{\partial x}(0, 0) = -1\), then

\[
F(x, c) = x \quad F(0, 0) + \frac{\partial F}{\partial x} x + \frac{\partial F}{\partial c} c = x - x + \frac{\partial F}{\partial c} c = x \quad x = \frac{1}{2} \frac{\partial F}{\partial c} c \quad (13.3)
\]

So, we have a line of fixed points \(x = X(c)\). (Note, that we cannot do it close to the other non-hyperbolic point with \(\frac{\partial F}{\partial x}(x^*, c^*) = 1\). Because in that case \(x\) at the left of the equation (13.3) will be canceled with the \(x\) at the right of the equation (13.3) and we will not find any solutions.)

Therefore if we introduce a new variable:

\[
y = x - X(c) \quad (13.4)
\]

we will get a new map

\[
y_{t+1} = f(y_t, c) \quad (13.5)
\]

which will have fixed points at \(y = 0\) for all \(c\):

\[
f(0, c) = 0 \quad (13.6)
\]

Therefore let us assume that we have a map with the following properties:

81
Let us now expand the right hand side of map (13.7) into the Taylor series.

### 13.1.2 Taylor expansion

Let us expand \( f(x, c) \) as a function of one variable \( x \). Because we have a parameter \( c \) the coefficients of our Taylor series will depend on this parameter:

\[
f(x, c) = f(0, c) + \frac{df}{dx}(0, c)x + \frac{d^2f}{dx^2}(0, c)\frac{x^2}{2} + \frac{d^3f}{dx^3}(0, c)\frac{x^3}{6} + \ldots
\]

here

\[
\phi = \frac{df}{dx}(0, c)
\]

\[
A = \frac{1}{2} \frac{d^2f}{dx^2}(0, c)
\]

\[
B = \frac{1}{6} \frac{d^3f}{dx^3}(0, c)
\]

Note, that \( \phi, A, B \) are functions of \( c \).

To remove \( Ax^2 \) we use the quadratic change of variables:

\[
x = y + ey^2
\]

We will also need an inverse change of variables up to the 3rd order, as we want to have a correct value of the new \( B \) coefficient. This inverse change of variables was found in the chapter on Hopf bifurcation in (7.20):

\[
y = x - ex^2 + 2e^2x^3
\]

Now normalization. First finding of \( y_{t+1} \)

\[
y_{t+1} = x_{t+1} - ex_{t+1}^2 + 2e^2x_{t+1}^3
\]

\[
= [\phi x_t + Ax_t^2 + Bx_t^3] - e[\phi x_t + Ax_t^2 + Bx_t^3]^2
\]

\[
+ 2e^2[\phi x_t + Ax_t^2 + Bx_t^3]^3
\]

note that up to the third order in \( x \):

\[
[\phi x_t + Ax_t^2 + Bx_t^3]^2 = \phi^2 x_t^2 + 2A\phi x_t^4 + 2Bx_t^4 + B^2 x_t^6 \approx \phi^2 x_t^2 + 2A\phi x_t^4
\]

\[
[\phi x_t + Ax_t^2 + Bx_t^3]^3 \approx \phi^3 x_t^3
\]

so we get:

\[
y_{t+1} = [\phi x_t + Ax_t^2 + Bx_t^3] - e[\phi^2 x_t^2 + 2A\phi x_t^4] + 2e^2[\phi^3 x_t^3]
\]

\[
= \phi x_t + (A - e\phi^2)x_t^2 + (B - 2Ae\phi + 2e^2\phi^3)x_t^3
\]
Now the direct change of variables: \( x = y + ey^2 \), note that up to the third order in \( x \):

\[
x^2 = (y + ey^2)^2 = y^2 + 2ey^3 + e^2y^4 \approx y^2 + 2ey^3
\]

\( x^3 \approx y^3 \)  

(13.15)

Hence:

\[
y_{t+1} = \phi y_t + (A + e\phi - e\phi^2)y_t^2 + \\
(B + 2Ae - 2Ae\phi - 2e^2\phi^2 + 2e^2\phi^3)y_t^3
\]

(13.16)

so

\[
e = \frac{A}{\phi^2 - \phi}
\]

(13.17)

We can always find \( e \) at \( c = 0 \).

From (13.9), and (13.7) we conclude, that \( \phi(0) = -1 \). Therefore at \( c = 0, e = A/2, \phi = -1 \)
and coefficient at \( y_t^3 \) is:

\[
d = B - 2Ae + 2Ae - 2e^2 = B + A^2 + A^2 - A^2/2 - A^2/2 = B + A^2
\]

(13.18)

From (13.9) we conclude, that at \( c = 0 \) as \( A = \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(0,0) \) and \( B = \frac{1}{6} \frac{\partial^3 f}{\partial x^3}(0,0) \), hence

\[
d = \frac{1}{6} \frac{\partial^3 f}{\partial x^3} + \frac{1}{4} \left( \frac{\partial^2 f}{\partial x^2} \right)^2
\]

(13.19)

So after this transformation our map becomes:

\[
y_{t+1} = \phi y_t + dy_t^3
\]

(13.20)

Note, that we can perform similar computation in order to remove a third order term \( x_{t+1} = \phi x_t + Bx_t^3 \) by \( x = y + fy^3 \). Similar computation results in \( f = \frac{B}{\phi^3 - \phi} \). Because at \( \phi(0) = -1, \phi^3 - \phi = 0 \), this third order term is non-removable as in case of the Hopf bifurcation.

By introducing new variable \( z = \sqrt{|d|} y \) we convert it to:

\[
z_{t+1} = \phi z_t \pm z_t^3
\]

(13.21)

Because in this case we expect a period two orbit, we will find a double iterated map. This is because a period two orbit is a fixed point of a double iterated map. We will find it also up to the third order in \( z \):

\[
z_{t+2} = \phi z_{t+1} \pm z_{t+1}^3
\]

\[
= \phi[\phi z_t \pm z_t^3] \pm [\phi z_t \pm z_t^3]^3
\]

\[
= \phi^2 z_t \pm [\phi + \phi^3] z_t^3 + ...
\]

(13.22)

Finally we will introduce a new parameter \( \mu \). For that, let use the Taylor expansion for \( \phi(c) = \phi(0) + \frac{\partial \phi}{\partial c}(0)c + ... \). Note, that from (13.9):
\[ \phi(c) = \frac{\partial f}{\partial c}(0,c) \]
\[ \phi(0) = \frac{\partial f}{\partial c}(0,0) = -1 \]
\[ \frac{d\phi}{dc}(0,0) = \frac{d^2 f}{dc^2} = \frac{\partial^2 f}{\partial x \partial c}(0,0) \]
\[ \phi(c) \approx -1 + \frac{\partial^2 f}{\partial x \partial c}c + \ldots \]

Hence it is reasonable to introduce \( \mu \) by replacing:

\[ \phi(c) = -1 \pm \mu \quad (13.24) \]

The `'+' sign corresponds to the positive value of \( \frac{\partial^2 f}{\partial x \partial c}(0,0) \), and the `'-' sign corresponds to the negative value of \( \frac{\partial^2 f}{\partial x \partial c}(0,0) \).

so our map (13.21) becomes

\[ z_{t+1} = -z_t \pm \mu z_t \pm z_t^3 \quad (13.25) \]

and our map (13.22) becomes:

\[ z_{t+2} = (-1 \pm \mu)^2 z_t \pm ((-1 \pm \mu) + (-1 \pm \mu)^3)z_t^3 \quad (13.26) \]

### 13.2 Study of the normal form

#### 13.2.1 Study of the normal form of a single iterated map

Let us study map (13.25) with \( d > 0, \frac{\partial^2 f}{\partial x \partial c}(0,0) > 0 \):

\[ z_{t+1} = -z_t + \mu z_t + z_t^3 \]

1. Fixed points.

\[ z = -z + \mu z + z^3 \quad 0 = z(-2 + \mu + z^2) \quad z = 0 \]

So we have one fixed point \( z = 0 \). (Note, that we can formally find other fixed points \( z = \pm \sqrt{2-\mu} \). But this fixed points are not good for us, as they are not close to \( z = 0 \). This is because for the very beginning we study the map around \( z = 0 \) only.)

2. Stability

\[ \frac{\partial f}{\partial z} = -1 + \mu + 3z = -1 + \mu \]

at \( \mu > 0 \) the point is stable, at \( \mu < 0 \) the point is unstable. So we get the following diagram. we see no further bifurcations here. To find other bifurcations we need to consider a double iterated map.

---

\[ \mu < 0 \quad \mu > 0 \]
13.3 Study of the normal form of a double iterated map

Let us transform the double iterated map (13.26).

First note, that we can rewrite this map as:

\[ z_{t+1} = (-1 \pm \mu)^2 z_t \pm ((-1 \pm \mu) + (-1 \pm \mu)^3) z_t^3 \]  

(13.27)

This just changes the numbering of the points \( z \), but does not change their values, dynamics, etc.

Now, we can easily see, that the function in the right hand side of (13.27) is odd: \( f(-z, \mu) = -f(z, \mu) \), and at \( z = 0, \mu = 0 \) this map has a non-hyperbolic fixed point: \( f(0, 0) = 0, \) \( \frac{\partial f}{\partial \mu}(0, 0) = 1 \). Therefore we expect a pitchfork bifurcation here studies in chapter 12.

Non-degeneracy conditions for the pitchfork bifurcation give:

\[ \frac{\partial^2 f}{\partial x \partial \mu}(0, 0) = \mp 2, \quad \frac{\partial^3 f}{\partial x^2 \partial \mu}(0, 0) = \pm 1. \]  

(13.29)

Therefore in accordance with the theorem map (13.27) can be transformed into the following normal form:

\[ Y_{t+1} = Y_t \mp \gamma Y_t \mp Y_t^3 \]  

(13.28)

Note, that from (12.8) it follows that:

\[ -\gamma \sim \frac{\partial^2 f}{\partial x \partial \mu}(0, 0) > 0 \quad \text{and} \quad \gamma \sim \frac{\partial^2 f}{\partial x \partial \mu}(0, 0) < 0 \]  

(13.29)

and that:

\[ -Y_t^3 \sim d(0, 0) = \left( \frac{1}{6} \frac{\partial^3 f}{\partial x^3} + \frac{1}{4} \left( \frac{\partial^2 f}{\partial x^2} \right)^2 \right) > 0 \quad \text{and} \quad Y_t^3 \sim d(0, 0) < 0 \]  

(13.30)

Double iterated map which corresponds to the case \( d > 0, \frac{\partial^2 f}{\partial x \partial \mu}(0, 0) > 0 \) is:

\[ Y_{t+1} = Y_t - \gamma Y_t - Y_t^3 \]  

(13.31)

Map (13.31) is a normal form for the pitchfork bifurcation. The bifurcation diagram for this case is shown at the left of the figure. However, this is a bifurcation diagram for a double iterated map. Its interpretation for the single iterate map is given in the left picture. The middle line is just a behavior of the equilibrium \( x = 0 \). However two bold gray lines become one orbit of the period two. This similar to the line of limit cycles in the Hopf bifurcation. The graphs of this map are shown in fig.13.3.

![Bifurcation diagrams for a double and single iterated map which undergoes flip bifurcation.](image-url)
13.3.1 Theorem. Flip (period doubling) bifurcation for maps.

Let \( x_{t+1} = f(x_t, c) \), has a non-hyperbolic fixed point \( x = 0, c = 0 \) such that:

\[
f(0, c) = 0 \quad \text{(for all } c \text{ close to } c = 0) \tag{13.32}
\]

\[
\frac{\partial f}{\partial x}(0, 0) = -1
\]

then if:

\[
\frac{\partial^2 f}{\partial x \partial c}(0, 0) \neq 0 \tag{13.33}
\]

\[
d = \frac{1}{6} \frac{\partial^3 f}{\partial x^3} + \frac{1}{4} \left( \frac{\partial^2 f}{\partial x^2} \right)^2 \neq 0 \tag{13.34}
\]

this map is locally equivalent to one of the following normal forms:

\[
x_{t+1} = -x_t \pm \gamma x_t + x_t^3 \tag{13.35}
\]

and the flip bifurcation takes place.

Note, that the sign at \( \gamma \) is the same as the sign of \( \frac{\partial^2 f}{\partial x \partial c} \), and the sign at \( x_t^3 \) is the same as the sign of \( d \).

Note, that condition (13.33) must be computed only after shifting the fixed point to the origin. This means, that in general we cannot apply this test for map (13.1). In order to be able apply it, we need first to transform our map to the form (13.5). This is because the transformation from (13.1) to (13.5) introduces \( y \) which is a function of \( c \), therefore derivatives \( \frac{\partial f}{\partial c} \) may be changed. For example, if we have a function of \( g(x, c) = x - c + c^2 \), then \( \frac{\partial g}{\partial c} = -1 + 2c \), however, if we denote \( y = x - c \), we get \( G(y, c) = y + c^2 \), and \( \frac{\partial G}{\partial c} = 2c \).
Figure 13.4: Bifurcation diagrams for the flip bifurcation
Chapter 14

Feigenbaum universality

14.1 Introduction

In this chapter we consider bifurcations of the logistic map. It will be shown that this map exhibits a complicated sequence of bifurcations which finally lead to a chaotic behavior. We will also show that there are some universal properties in the bifurcation sequence which leads to chaos. This universality is called the Feigenbaum universality.

We will start with some simple properties of the logistic map.

Consider the map:

\[ x_{t+1} = \lambda x_t (1 - x_t) = f(x_t, \lambda) \]  

(14.1)

The maximum of the right hand side here is:

\[ f_{\text{max}} = \frac{\lambda}{4} \]  

(14.2)

If \( \lambda > 1 \) we have two fixed points in the interval \( 0 \leq x \leq 1 \). If we now consider an initial point \( x_0 \) from the interval \( 0 \leq x \leq 1 \), then the maximal possible value of the next \( x_1 \) will be

\[ x_1 = f(x_0, \lambda) \leq f_{\text{max}} = \frac{\lambda}{4} \]  

(14.3)

Therefore, if \( \lambda < 4 \), then \( x_1 \) will be also in the interval \( 0 \leq x \leq 1 \). Similarly \( x_2 \) will be in the interval \( 0 \leq x \leq 1 \), etc. So all orbit \( x_t \) will be bounded in the interval \( 0 \leq x \leq 1 \).

What behavior do we have in this interval? The first fixed point

\[ x_{f1} = 0 \]

in unstable at \( \lambda > 1 \). The second fixed point is

\[ x_{f2} = \frac{\lambda - 1}{\lambda} \]

This point is stable for \( \lambda \leq 3 \). At \( \lambda = 3 \) we have the flip bifurcation. After that the fixed point \( x_{f2} \) becomes non-stable and the only stable limit orbit here will be the orbit of period two.

14.1.1 The phenomenon

It turns out that this period doubling bifurcation is just a beginning of the sequence of further period doubling bifurcations. The bifurcation diagram is shown in the figure. It has the following properties:
There is an increasing sequence of parameter values \( \lambda_1 < \lambda_2 < \lambda_3 \ldots \) at which the logistic map repeatedly undergoes a period doubling bifurcation. This means that at \( \lambda = \lambda_2 \) the orbit of the period 2 becomes unstable and a stable periodic orbit of twice the period (i.e., 4) becomes stable. Then period 4 orbit bifurcates into the stable period 8 orbits, etc.

It turns out that the sequence \( \lambda_k \) is the converging sequence with the limit

\[
\lim_{k \to \infty} \lambda_k = \lambda_\infty = 3.570
\]  

(14.4)

3. The convergence is exponential. i.e.,

\[
\lim_{k \to \infty} \frac{\lambda_k - \lambda_{k-1}}{\lambda_{k+1} - \lambda_k} = 4.6692
\]  

(14.5)

i.e., the distance between the successive bifurcations decreases.

4. The limit 4.6692 is a universal constant for a large class of maps \( x_{t+1} = f(x_t, \lambda) \)

We will study the phenomenon of Feigenbaum universality using two approaches: qualitative and quantitative.

The main aim of the qualitative approach is to understand why the cascade of the bifurcations occurs. In qualitative approach we will study why this phenomenon is universal.

### 14.2 Qualitative approach

Our plan here will be the following:

1. One branch. Here we will find how we study bifurcations diagrams which contain a lot of branches (see fig1). Our main results will be that we can study just one branch of the diagram, and later will draw the whole bifurcation diagram from that one branch.

2. Study of the first flip bifurcation

3. Show the similarity of the first and of the second flip bifurcation. Explain why the successive flip bifurcations occur.
Let us start our study.

1. One branch. Let us start with representation of a bifurcation diagram for the first flip bifurcation. After this bifurcation we get two branches a,b on our diagram, but they represent the same orbit of period 2.

![Bifurcation Diagram](image)

This means that although at each parameter value we have two points, these two points will bifurcate simultaneously at the same parameter value. To prove it let us consider the hyperbolicity of this period 2 orbit. As we know a period two orbit is a fixed point of the double iterated map. The hyperbolicity is determined by the derivative of this map. We get:

\[ x_2 = f(x_1); \quad x_1 = f(x_2) \quad (14.6) \]

The derivative at each point is:

\[ (f^{(2)}(x_0))' = (f(f(x_0)))' = f'(f(x_0))f'(x_0) \quad (14.7) \]

In our case due to condition (14.6)

\[ (f^{(2)}(x_1))' = f'(f(x_1))f'(x_1) = f'(x_2)f'(x_1) \quad (14.8) \]

similarly

\[ (f^{(2)}(x_2))' = f'(f(x_2))f'(x_2) = f'(x_1)f'(x_2) \quad (14.9) \]

Therefore

\[ (f^{(2)}(x_1))' = (f^{(2)}(x_2))' \quad (14.10) \]

and the behavior of the both branches will be the same. They will both become unstable and will both bifurcate at the same parameter value.

This means that instead of studying all the branches we can pickup just one branch and follow it (see fig.2b). Later we can recover all the branches from that one.

In general we can choose any branch. Let us pickup the branch which is closest to the point 0.5.

2. Study of the first flip bifurcation. To study flip bifurcation we need to find fixed points of the double iterated map. The map which is the double iterated for the logistic map is:

\[ x_{r+2} = \lambda x_{r+1}(1 - x_{r+1}) = \lambda (\lambda x_r(1 - x_r)(1 - \lambda x_r(1 - x_r))) = \lambda^2 x_r(1 - x_r)(1 - \lambda x_r + \lambda x_r^2) \quad (14.11) \]
Its graph is shown in fig.3 We see that when $\lambda < 3$ (the left figure) we have two fixed points on the double iterated map which correspond to the fixed points in our original map. When $\lambda > 3$ we get two extra fixed points. But both they correspond to the one period two orbit. (The middle of the figure). If we further increase $\lambda$ one of the fixed points moves to the left.

Figure 14.3: Changes in the shape of the double iterated map at change of the parameter $\lambda$

3. Similarity of the first and second flip bifurcations.

Figure 4 shows the changes in graph of the double iterated map when we increase the parameter $\lambda$. Let us concentrate our attention to the fixed point located in the dashed square. We see that when we increase the parameter value form $\lambda = 3.2$ to $\lambda = 3.46$ the fixed point which is located inside this square moves.

Figure 14.4: Double iterated map at $\lambda = 3.2$ left; and $\lambda = 3.46$ right picture

Let us do the following. Let us cut this square and rotate it on 180 degrees. In that case we will get the graphs which are shown in fig.5.

We see that the changes which occur inside the dashed square is the similar to the changes in functions which occur during the flip bifurcation for a single iterated map. The slope of the graph at the fixed point increases and becomes equal to $-1$. Then we have a flip bifurcation in which this fixed point becomes unstable and we get a stable period two orbit. But in this figure we displayed a part of a double iterated map! Therefore here a period 2 orbit becomes unstable and we get a stable period 4 orbit. However we can study this bifurcation of a period 2 orbit on a map which is a double iterated map of the map which is displayed in this figure. Therefore we again will get figure 3, and then figure
4,5 and finally we will get one more flip bifurcation. If we go further with this process we will get an infinite sequence of such period doubling bifurcations which are presented at the bifurcation diagram in fig1.

14.3 Quantitative approach

Our plan here will be the following:

1. Universality with respect to the parameter change
2. Doubling operator
3. Period doubling in an infinite dimensional functional space

Let us start our study.

1. Universality with respect to the parameter change. Sometimes it seems that it is impossible to have universal constants in such systems with parameter because we can always change our parameter. Say we denote $\lambda^3 + 2$ as a new parameter. Why do we expect that this change will not affect limit (14.5)?

It turns out that the phenomenon is universal because it represented in a special universal form which takes care about all possible “good” parameter changes.

Let us prove it. Assume that we have a sequence:

$$\lim_{k \to \infty} \lambda_k = \lambda_\infty$$

with

$$\lim_{k \to \infty} \frac{\lambda_k - \lambda_{k-1}}{\lambda_{k+1} - \lambda_k} = C$$

Let us introduce a new parameter

$$\mu = g(\lambda)$$

Let us find the value of the constant $C$ for the new parameter $\mu$.

Because $\mu = g(\lambda)$,

$$\lim_{k \to \infty} \mu_k = \lim_{k \to \infty} g(\lambda_k) = g(\lim_{k \to \infty} \lambda_k) = g(\lambda_\infty)$$
Because our sequence converges to the value \( g(\lambda_\infty) \) the points \( g(\lambda_k) \) will be close to the value \( g(\lambda_\infty) \) for sufficiently large \( k \). Therefore, in order to study our system at large \( k \) let us expand our function \( g(\lambda) \) into the Taylor series close to its limit value:

\[
  g(\lambda) \approx g(\lambda_\infty) + g'(\lambda_\infty)(\lambda - \lambda_\infty)
\]  

(14.12)

or

\[
  \mu_k = g(\lambda_k) \approx g(\lambda_\infty) + g'(\lambda_\infty)(\lambda_k - \lambda_\infty)
\]  

(14.13)

\[
  \mu_{k-1} = g(\lambda_{k-1}) \approx g(\lambda_\infty) + g'(\lambda_\infty)(\lambda_{k-1} - \lambda_\infty)
\]  

(14.14)

By subtracting the equation (14.14) from (14.13) we get:

\[
  \mu_k - \mu_{k-1} = g(\lambda_k) - g(\lambda_{k-1}) \approx g'(\lambda_\infty)(\lambda_k - \lambda_{k-1})
\]  

(14.15)

Similarly:

\[
  \mu_{k+1} - \mu_k = g(\lambda_{k+1}) - g(\lambda_k) \approx g'(\lambda_\infty)(\lambda_{k+1} - \lambda_k)
\]  

(14.16)

Therefore:

\[
  \frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k} = \frac{g(\lambda_k) - g(\lambda_{k-1})}{g(\lambda_{k+1}) - g(\lambda_k)} \approx \frac{g'(\lambda_\infty)(\lambda_k - \lambda_{k-1})}{g'(\lambda_\infty)(\lambda_{k+1} - \lambda_k)} = \frac{\lambda_k - \lambda_{k-1}}{\lambda_{k+1} - \lambda_k} = C
\]  

(14.17)

Therefore if we change the parameter we will get the same limit value of \( C \). Therefore we have proved that the constant \( C = 4.6692 \) from equation (14.5) is universal with respect to the parameter change.

2. Doubling operator. The idea of doubling operator is formal representation of the process of transformation of pictures which occurs in fig.4,5. The main idea here is to write formally the transformation of a single iterated map into the double iterated map, then finding the part of this double iterated map in a dashed square, cutting and rotation of the map and finally getting the graph which looks similar to the initial single iterated map.

Formally we have a single iterated map which is given by the equation

\[
  x_{t+1} = f(x_t)
\]  

(14.18)

We make the following transformations:

(a) Double iterated map. This is just \( f^{(2)}(x) \).

(b) Rotate for 180°. This is just

\[
  F_{\text{new}} = -f(-x)
\]  

(14.19)

Here the first “minus” at our function changes the direction of the y-axis, so it is just reflection, the second “minus” at \( x \) changes the direction of the x axis, i.e., our fixed point which was at the upper right corner of fig.4 after such transformation will be at the lower left corner.
(c) Scale our map that it has the same spatial size as a former single iterated map. We want that \( x \) and \( y \) in a new map will range from 0 to 1. For that we need to stretch our small parabola from fig.4 in the \( x \) and the \( y \) directions. This stretch is given by:

\[
x_{\text{old}} = \frac{x_{\text{new}}}{\alpha} \quad \text{or} \quad F_{\text{new}} = f\left(\frac{x_{\text{new}}}{\alpha}\right)
\]

(14.20)

If our old variable was changing in a region \( 0 < x_{\text{old}} < 0.1 \), then by choosing \( \alpha = 10 \) we get a new variable which will change in the region \( 0 < x_{\text{new}} < 1.0 \), as \( x_{\text{new}} = \alpha \times x_{\text{old}} \).

Similarly stretch in the \( y \) direction is:

\[
y_{\text{old}} = \frac{y_{\text{new}}}{\alpha} \quad \text{or} \quad y_{\text{new}} = y_{\text{old}} \times \alpha = f(x) \times \alpha
\]

(14.21)

(d) Conclusion. All these operations together give us the following transformation for our function

\[
f_{\text{new}} = -\alpha f^{(2)}\left(-x/\alpha\right) = T f
\]

(14.22)

The operator \( T \) is so called functional operator. It acts on functions. Because each function consists of infinite number of points, this operator takes infinite number of points and produces another infinite number of points. So it is infinite-dimensional operator, and the map:

\[
f_{k+1} = T f_k = -\alpha f^{(2)}\left(-x/\alpha\right)
\]

(14.23)

is an infinite dimensional map.

One way to imagine the axis in that infinite dimensional space is to represent a function as its Taylor series. Then use infinite number of axes with axis 1 for the value of the first coefficient of Taylor series \( a_1 = df/dx \), the second axis is for the value of the coefficient \( a_2 \), etc. As we have infinite number of coefficients we will get infinite number of axis. The function in such space will be just a point with coordinates \( a_1, a_2, a_3, \ldots \).

3. Period doubling in a infinite-dimensional functional space.

Now consider the map

\[
f_{k+1} = T f_k
\]

(14.24)

As usual map it can have a fixed point which satisfies:

\[
f^* = T f^*
\]

(14.25)

Of course this fixed point is a function. This function was computed using numerical methods and the first terms in Taylor series for this function were the following:

\[
f = 1 - 1.52763x^2 + 0.104815x^4 + \ldots \quad \alpha = 2.50290
\]

(14.26)

When we know the fixed point we can study its stability. It turns out that we can find that this point is a saddle point, i.e., it is non-stable. However this saddle point has one important property. First, because we consider functions in infinite dimensional space this saddle point has infinitely many manifolds. However the amazing fact is that all but one manifolds are stable and we have only one unstable manifold.

Now let us draw period doubling cascade in that space. Fist let us consider all functions which have the flip bifurcation. The set of these functions will give us a hyper-plane in
our infinite-dimensional space. To see it let as assume that we use the coefficients of Taylor series as axis of our coordinate system. Then the condition for the flip bifurcation is \( \frac{\partial f}{\partial x}(x^*) = -1 \). This means that one of the coefficients of the Taylor series is \(-1\), or that one of the coordinates is fixed at the value \(-1\). This will give us a plane parallel to the origin plane. (For example if we consider three dimensional space the \(x, y, z\), then the condition \(z = -2\) will give the plane which is parallel to the \(xy\) plane.) Let us denote this plane as 1 in fig.6. So any function \(f\) form plane 1 has flip bifurcation.

Now consider the following expression:

\[
f = Tg = -\alpha g^{(2)}(-x/\alpha)
\]

(14.27)

where \(f\) belongs to plane 1. As \(f\) has a flip bifurcation and we get a stable period 2 orbit, then the function \(g\) will have a flip in which we get a stable period 4 orbit. This is because from (14.27) \(f\) is a double iterated map of \(g\). And therefore the fixed point of \(f\) is the period 2 orbit of \(g\) and the period 2 orbit of \(f\) is the period 4 orbit of \(g\). So the functions:

\[
g = T^{-1}f
\]

(14.28)

have period 4 orbits. Let us draw the set of all functions \(g\). Similarly we will get a plane 2. All functions from this plane have flip bifurcations in which they get a stable period 4 orbits.

\[
h = T^{-1}g
\]

(14.29)

will give us a plane of stable period 8 orbits, etc, so by applying the doubling operator we will finally get a plane where the functions have period infinity orbits. Where is that plane infinity located?

If we have a non-stable manifold as in fig.7, then our forward map will drive our system to infinity. However the inverse map where we go \(x_4 \rightarrow x_3 \rightarrow x_2\ldots\) will bring us to the fixed point. Now, in case fig.6 we have only one unstable manifold. Therefore the inverse
map $T^{-1}$ will move our planes back and finally we will arrive to the limit plane which is the plane of stable manifolds.

Now about functions with parameter. If we have a map which depends on a parameter, e.g. the logistic map, then we have $f(x, \lambda)$, therefore it will give us a line in the space of our functions $f$. It is very probable that this line will cross all the planes including the final plane of stable manifolds. What will be the dynamics? The dynamics will be such. As the line of functions $f(x, \lambda)$ crosses the plane $1$ we will observe the first period doubling. When it crosses plane $2$ we will get the second period doubling, etc. And when it will cross the plane infinity we will get an infinite period orbit in our map. So we see why these cascades occur. Now, why do they have a universal scaling?

To see it let us consider unstable manifold. It is a line, so we can introduce coordinates along this line. Our idea will be to use these coordinates as a new parameter. We can do it in the following way. When our line of functions showed as a thin line in fig.8 will cross the plane $1$, it will have a point, and this point will have some coordinate along the non-stable manifold. Let us denote it as $\eta_1$. In a similar way the point of intersection with the second plane will have some coordinate $\eta_2$ etc. What is dynamics of these coordinates. The dynamics along the non-stable manifold is given by:

$$\eta_{k+1} = \delta \eta_k$$ (14.30)

where $\delta$ is the eigen value of the map $T$. What will be behavior for the inverse map $T^{-1}$?

It will be

$$\eta_{k+1} = \delta^{-1} \eta_k$$ (14.31)

So the value of the expression (14.5) for the parameter $\eta$ coordinates will be:

$$\frac{\eta_k - \eta_{k-1}}{\eta_{k+1} - \eta_k} = \frac{\eta_k - \eta_k \delta}{\eta_k \delta^{-1} - \eta_k} = \frac{1 - \delta}{\delta^{-1} - 1} = \delta = 4.6692$$ (14.32)

So we proved universality for the parameter $\eta$. However if we will use other parameter ($\lambda$) it will not change the limit (14.32). We have proved it at the beginning of this section.

Now, why it is universal for many different maps? This is because this phenomenon is general for all function for which the operator $T$ can be introduced. so it is valid not only for the logistic map but for many other maps with similar properties.

Note, that sometimes the line of functions can go back before reaching the plane infinity. In that case we will observe just several period doubling bifurcation as it was in the Bier Bountis model from tutorials on 1D maps.
Figure 14.8:
Chapter 15

Maps in 2D

15.1 Linear maps

15.1.1 Real eigen values

Consider a general linear planar map:

\[
\begin{align*}
x_{n+1} &= ax_n + by_n \\
y_{n+1} &= cx_n + dy_n
\end{align*}
\]

(15.1)

It is possible to prove that the general solution of this map is:

\[
\begin{pmatrix} x_n \\ y_n \end{pmatrix} = C_1 \begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} \lambda_1^n + C_2 \begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix} \lambda_2^n
\]

(15.2)

where \( \lambda_1, \lambda_2 \) are eigen values of the matrix

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]

and \( \begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix}, \begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix} \) are the corresponding eigen vectors.

Note, that the prove of formulae (15.2) is similar to the prove of the formulae for the general solution of 2D ODEs and it follows from the obvious fact, that if we have a linear 1D map \( x_{n+1} = ax_n \), then for any initial value of \( x = x_0 \) the solution will be \( x_n = x_0 a^n \).

The formula (15.2) is valid for real as well as for complex eigen values. However, in case of the complex eigen values we need to extract a real part of the expression (15.2). We will do in the next section.

15.1.2 Complex eigen values

If eigen values are complex:

\[ \lambda_{1,2} = \alpha \pm i\beta = |\lambda|e^{\pm i\phi} \]

(15.3)

we can apply the same procedure of finding of the real part of this solution as for ODEs. The main part of that procedure was using of Euler representation for \( e^{\lambda t} \):

\[
e^{\lambda t} = e^{(\alpha + i\beta)t} = e^{\alpha t} (\cos \beta t + i \sin \beta t)
\]

(15.4)

Similar formulae for maps gives:

\[ (\alpha + i\beta)^n = |\lambda|^n e^{i\phi n} = |\lambda|^n (\cos \phi n + i \sin \phi n) \]

(15.5)
So if we make steps similar to the case of complex eigen values of ODEs we will get the following formulae for the general solution of the map with complex eigen values:

\[
\begin{pmatrix} x_n \\ y_n \end{pmatrix} = C_1 y_1 + C_2 y_2
\]

where \( C_1 \) and \( C_2 \) are arbitrary constants, and:

\[
y_1 = |\lambda|^n (v_r \cos \phi n - v_i \sin \phi n) \\
y_2 = |\lambda|^n (v_r \sin \phi n + v_i \cos \phi n)
\]

where \( v_r, v_i \) are the real and complex parts of the eigen vector of map (15.1):

\[
\begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} = v_r + iv_i
\]

### 15.2 Nonlinear maps

Consider a general nonlinear planar map:

\[
\begin{align*}
x_{t+1} &= f(x_t, y_t) \\
y_{t+1} &= g(x_t, y_t)
\end{align*}
\]

Fixed points of this map are given by the conditions:

\[
\begin{align*}
x^* &= f(x^*, y^*) \\
y^* &= g(x^*, y^*)
\end{align*}
\]

We can linearize our map close to equilibria using Taylor series. For that we first shift the equilibrium to the point \((0, 0)\) and linearize the right hand sides of our map. We find the following:

\[
\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = J \begin{pmatrix} x_t \\ y_t \end{pmatrix}
\]

where \( J \) is the Jacobian of our map

\[
J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}
\]

The Jacobian has two eigen values which sometimes are called multipliers. The stability of equilibrium is determined by eigen values of the Jacobian. It is easy to see from the general solution of 2D maps (15.2,15.6,15.7) that if \(|\lambda| < 1\), then \(x\) and \(y\) are decreasing with increase of \(n\) and a fixed point is stable. This result has a nice geometrical image given in the following theorem.

**Theorem 13** Assume that linearization (15.11) of map (15.9) has two eigen values which can be both real \(\lambda_1, \lambda_2\), or complex conjugate \(\lambda_{1,2} = \alpha \pm i\beta\). The fixed point of the map (15.9) is stable if the eigen values of the Jacobian are inside the unit circle on the \((\alpha, \beta)\) plane.
To explain the theorem, note, that modulus of a complex number $|\lambda| = \sqrt{\alpha^2 + \beta^2}$, and this is just a distance from a point $(\alpha, \beta)$ on the $(\alpha, \beta)$ plane to the origin (see fig. 15.1). Therefore, the condition of stability $|\lambda| < 1$, is equivalent to the condition that the both $\lambda$ are inside the unit circle on the $(\alpha, \beta)$ plane. Note, that if $\lambda$ is a real number, then we can view it as a complex number $\alpha + i\beta$ with $\beta = 0$, and therefore the condition for stability that both $|\lambda| < 1$, still means that both $\lambda$ are inside the unit circle on the $(\alpha, \beta)$ plane, however, in this case they are located at the $\alpha$ axis only.

As in previous cases stability is closely connected with hyperbolicity:

**Definition 17** A fixed point of the map (15.9) is said to be hyperbolic if linear map (15.11) has no eigenvalues with modulus 1

And again in hyperbolic case linear and non-linear systems are equivalent to each other.

**Theorem 14** Let $x^*$ be a hyperbolic fixed point of the map (15.9). Then there is a neighborhood $X^*$ of $x^*$ and a neighborhood of the origin $N$ such that the map (15.9) in $X^*$ is equivalent to its linear map (15.11) in $N$.

Similar theorem exists for the system with parameter:

$$
x_{t+1} = f(x_t, y_t, c) \\
y_{t+1} = g(x_t, y_t, c)
$$

(15.13)

Therefore we do not expect any bifurcations close to hyperbolic equilibria. Let us study non-hyperbolic cases. We can have two different non-hyperbolic cases:

1. If eigenvalues of the Jacobian are real, and $\lambda_1 = \pm 1; \lambda_2 \neq \pm 1$;

2. If eigen values are complex: $\lambda_{1,2}(c) = \alpha(c) \pm i\beta(c) = |\lambda(c)| e^{i\phi(c)}$; \[|\lambda(0)| = \sqrt{\alpha(0)^2 + \beta(0)^2} = 1.\]

In case 1, as for 1D maps we will get fold, transcritical, pitch-fork or flip bifurcations but now on a center manifold of our map.

In case 2 we will get a new bifurcation which is similar to Hopf bifurcation of ODEs. We will start with a study of this second case of “Hopf” bifurcation which is called the “Neimark-Sacker” bifurcation.
Chapter 16
Neimark-Sacker bifurcation

Consider a planar map:

\[ x_{t+1} = f(x_t, y_t, c) \]
\[ y_{t+1} = g(x_t, y_t, c) \]  

(16.1)

Assume, that is has a non-hyperbolic fixed point at \( x = 0, y = 0, c = 0 \) with complex eigenvalues such that:

\[ \lambda_{1,2}(c) = \alpha(c) \pm i\beta(c) = |\lambda(0)| e^{i\phi(c)}; \quad |\lambda(0)| = \sqrt{\alpha(0)^2 + \beta(0)^2} = 1 \]  

(16.2)

We will study bifurcations which occur around this non-hyperbolic fixed point of this map (16.1). Our first step will be simplification of our system using Maclaurin series.

### 16.0.1 Linear terms.

If we perform the same steps as we did for system of two ODEs around the point of Hopf bifurcation in section 7.1.1 we will transform nonlinear map (16.1) to a complex map:

\[ z_{t+1} = \lambda z_t + F(z, \bar{z}, c) \]  

(16.3)

where \( z = x + iy \), and \( F(z, \bar{z}, c) \) is a function which contains terms of the order \( z^2, \bar{z}^2 \) and higher.

### 16.0.2 nonlinear terms

Similarly the Maclaurin approximation of (16.3) is given by:

\[ z_{t+1} = \lambda z_t + \frac{\partial^2 F}{\partial z^2} z_t^2 + \frac{\partial^2 F}{\partial z \partial \bar{z}} z_t \bar{z} + \frac{\partial^2 F}{\partial \bar{z}^2} \bar{z}_t^2 \]

\[ + \frac{\partial^3 F}{\partial z^3} z_t^3 + \frac{\partial^3 F}{\partial z^2 \partial z} z_t^2 \bar{z}_t + \frac{\partial^3 F}{\partial z \partial \bar{z}^2} z_t \bar{z}_t^2 + \frac{\partial^3 F}{\partial \bar{z}^3} \bar{z}_t^3 \ldots \]  

(16.4)

The procedure of removing of nonlinear terms here is similar to the procedure of removing of nonlinear terms for the Hopf bifurcation considered in chapter 7. Let us illustrate it on example of one quadratic term:

\[ z_{t+1} = \lambda z_t + A z_t^2 \]  

(16.5)

It turns out that we can remove this term by the following quadratic change of variables:

\[ z = w + aw^2 \]  

(16.6)
As we computed in (7.21) the inverse transformation is given by:

\[ w \approx z - az^2 \]  

(16.7)

The formal substitution is just a modification of equations (7.23, etc) and can be done as follows. First we find the dynamics of the new variable \( w \):
he first step is finding dynamics of the new variable \( w \):

\[ w_{t+1} = z_{t+1} - az_{t+1}^2 \]  

(16.8)

Now, we have to replace \( z_{t+1} \) by its expression (16.5), we find:

\[ w_{t+1} = \lambda z_t + Az_t^2 - a(\lambda z_t + Az_t^2)^2 = \lambda z_t + (A - \lambda^2 a)z_t^2 + O(|z|^3) \]  

(16.9)

Now we need to replace \( z_t \) using the direct transformation (16.6):

\[ w_{t+1} = \lambda(w_t + aw_t^2) + (A - \lambda^2 a)(w_t + aw_t^2)^2 + O(|w|^3) = \lambda w_t + \lambda aw_t^2 + (A - \lambda^2 a)(w_t^2 + 2aw_t^3 + a^2w_t^4) + O(|w|^3) \]  

Note, that \( O(|z|^3) \) naturally becomes \( O(|w|^3) \) as any term of the third order in \( z^3 \) will be obviously of the third order in \( w \) as \( (w + aw)^3 = O(|w|^3) \). Note also, that we can put other terms like \( 2aw^3, a^2w^4 \) to \( O(|w|^3) \) yielding:

\[ w_{t+1} = \lambda w_t + w_t^2(\lambda a + A - \lambda^2 a) + O(|w|^3) = \lambda w_t + w_t^2(A + (\lambda - \lambda^2)a) + O(|w|^3) \]  

Therefore if we choose

\[ a = \frac{A}{\lambda^2 - \lambda} \]

then our transformation (16.6) will remove the quadratic term in (16.5). We can do it provided the denominator in the above equation is not zero, i.e.,

\[ \lambda^2 - \lambda \neq 0 \lambda(\lambda - 1) \neq 0 \lambda \neq 1 \]

\[ z = \lambda z + Az^2 \]. Of course this transformation can change cubic terms which are collected in \( O(|w|^3) \). However, they are of the higher order and we can handle them latter.

In a similar way, we can remove the following non-linear terms:

\[ z_{t+1} = \lambda z_t + Bz_t \tilde{z}_t \ by \ z = w + bw \tilde{w} \ with \ b = \frac{B}{\lambda^2 - \lambda} \ if \ \lambda^2 \neq 1 \]

\[ z_{t+1} = \lambda z_t + Cz_t^2 \ by \ z = w + cw \ tilde{w} \ with \ c = \frac{C}{\lambda^2 - \lambda} \ if \ \lambda \neq 1 \]

\[ z_{t+1} = \lambda z_t + Dz_t^3 \ by \ z = w + dw^3 \ with \ d = \frac{D}{\lambda^3 - \lambda} \ if \ \lambda^2 \neq 1 \]

\[ z_{t+1} = \lambda z_t + Fz_t^2 \tilde{z}_t \ by \ z = w + f\tilde{w}^2 \ with \ f = \frac{F}{\lambda^2\lambda - \lambda} \ if \ \lambda^2 \neq 1 \]

\[ z_{t+1} = \lambda z_t + Gz_t^3 \tilde{z}_t \ by \ z = w + g\tilde{w}^3 \ with \ g = \frac{G}{\lambda^4 - \lambda} \ if \ \lambda^4 \neq 1 \]  

(16.10)

Note, that conditions on \( \lambda \) in the above formulae were obtained for \( c = 0 \) using the following properties of \( \lambda(c) \): at \( c = 0 \) (16.2): \( |\lambda(0)|^2 = \lambda \tilde{\lambda} = 1 \), i.e., \( \lambda(0) = \frac{1}{\lambda(0)} \).

However there will be one non-removable term. If we would like to remove in a similar way the term:

\[ z_{t+1} = \lambda z_t + Ez_t^2 \tilde{z}_t \ by \ z = w + ew \tilde{w} \]

\[ z_{t+1} = \lambda z_t + Ez_t^2 \tilde{z}_t \ by \ z = w + ew \tilde{w} \]
We will find that after such transformation our equation becomes:

\[ w_{t+1} = \lambda w_t + w_t^2 \tilde{w}_t (E - e(\lambda - \lambda^2)) \]

However, because at the bifurcation point \( c = 0 \), \( \lambda(0) = 0 \), \( e(\lambda - \lambda^2) = e(\lambda - \lambda) = 0 \) for any \( e \), and the term \( w_t^2 \tilde{w}_t (E - e(\lambda - \lambda^2)) \) persists for any value of \( e \), i.e., it is not removable.

Our final conclusion is, that any system (16.3) can be transformed to the form:

\[ w_{t+1} = \lambda w_t + \delta w_t^2 \tilde{w}_t \]  

By using polar representations of \( \lambda = |\lambda| e^{i\phi} \) and \( w = r e^{i\theta} \) after some transformations it is possible to get the following normal form in polar coordinate system:

\[ r_{t+1} = |\lambda| r_t + d r_t^3 \]
\[ \theta_{t+1} = \theta_t + \phi(c) + b(c) r_t^2 \]  

16.1 Study of the normal form

We can have two different cases: \( d > 0 \), or \( d < 0 \).

16.1.1 Case \( d < 0 \).

Fixed points, equation for \( r \)

\[ r^* = |\lambda| r^* + d r^* \]
\[ r^* = 0 \]
\[ 1 = |\lambda| + d r^* \]
\[ \lambda > 1 \]

Stability:

\[ df/dr = |\lambda| + 3dr^2 \]
\[ r = 0; \quad df/dr = |\lambda| \text{ stable for } |\lambda| < 1 \text{ unstable for } |\lambda| > 1 \]  
\[ r = \sqrt{\frac{1-|\lambda|}{d}}; \quad df/dr = |\lambda| + 3d \frac{1-|\lambda|}{d} = 3 - 2|\lambda| \text{ stable as } |\lambda| > 1 \]  

We have the following bifurcation diagram for \( r \), and the following dynamics in 2D:

![Diagram](image)

Figure 16.1: Diagram for the Neimark-Sacker bifurcation for the case \( d < 0 \).
16.1.2 Case $d > 0$.

We have the following bifurcation diagram for $r$, and the following dynamics in 2D:

![Diagram for the Neimark-Sacker bifurcation for the case $d > 0$.](image)

Figure 16.2: Diagram for the Neimark-Sacker bifurcation for the case $d > 0$.

16.2 Theorem. Neimark-Sacker bifurcation.

Let a 2D map

\[
\begin{align*}
    x_{t+1} &= f(x_t, y_t, c) \\
    y_{t+1} &= g(x_t, y_t, c)
\end{align*}
\]  \hspace{1cm} \text{(16.15)}

has a non-hyperbolic fixed point at $x = 0, y = 0, c = 0$, such that eigen values of the Jacobian matrix are:

\[
    \lambda = |\lambda(c)| e^{i\phi(c)}, \quad |\lambda(0)| = 1
\]  \hspace{1cm} \text{(16.16)}

then if

\[
    \frac{\partial |\lambda|}{\partial c}(0) \neq 0; \quad e^{ik\phi(0)} = \lambda^k \neq 1 \ (k = 1, 2, 3, 4); \quad d(0) \neq 0
\]  \hspace{1cm} \text{(16.17)}

there is a coordinate change which transforms (16.1) into the following form:

\[
\begin{align*}
    r_{t+1} &= |\lambda| r_t + d r_t^3 \\
    \theta_{t+1} &= \theta_t + \phi(c) + b(c) r_t^2
\end{align*}
\]  \hspace{1cm} \text{(16.18)}

and the Neimark-Sacker bifurcation takes place.

Note, that this normal form has the following representation in the Cartesian coordinate system:

\[
\begin{pmatrix}
    x_{t+1} \\
    y_{t+1}
\end{pmatrix}
= \begin{pmatrix}
    \cos \phi & -\sin \phi \\
    \sin \phi & \cos \phi
\end{pmatrix}
\begin{pmatrix}
    r_t \\
    \theta_t
\end{pmatrix}
+ |\lambda| \begin{pmatrix}
    x_t \\
    y_t
\end{pmatrix}
+ a b \begin{pmatrix}
    a & -b \\
    b & a
\end{pmatrix}
\begin{pmatrix}
    x_t \\
    y_t
\end{pmatrix}
+ O(||x||^4).
\]  \hspace{1cm} \text{(16.19)}

and $d(0) = a(0)$
Chapter 17

Center manifold for maps

17.1 Main theorems

The center manifold theory for maps is similar to the center manifold theory for the ODEs. First consider a map without parameters. This map can be transformed to the map in which the matrix of linear terms is in the Jordan normal form. Therefore if the eigen values of our map are real the map will be:

\[ \begin{align*}
  x_{t+1} &= \lambda_1 x_t + f(x_t, y_t) \\
  y_{t+1} &= \lambda_2 y_t + g(x_t, y_t)
\end{align*} \] (17.1)

Here \( f, g \) are the functions of the second order in \( x_t \) and \( y_t \), i.e., they do not have any linear terms in their Taylor series.

If we consider a linearization of system (17.1) we get a system:

\[ \begin{align*}
  x_{t+1} &= \lambda_1 x_t \\
  y_{t+1} &= \lambda_2 y_t
\end{align*} \] (17.2)

System (17.2) has the phase portrait which includes a stable manifold \( E^s \) if \( |\lambda| < 1 \), a non-stable manifold \( E^u \) if \( |\lambda| > 1 \), or a central manifold \( E^c \) if \( |\lambda| = 1 \).

As for the ODEs there is a theorem which establishes similar manifolds in a non-linear system (17.1)

**Theorem 15**

1. Let \( x = 0, y = 0 \) is a fixed point of the map

\[ \begin{align*}
  x_{t+1} &= F(x_t, y_t) \\
  y_{t+1} &= G(x_t, y_t)
\end{align*} \] (17.3)

Assume that linearization of this map has one eigen value \( |\lambda_1| = 1 \) and the other \( |\lambda_2| \neq 1 \). Hence the linear system has a central manifold \( E^c \) and stable (non-stable) manifold \( E^c, (E^u) \).

Then there exist central \( W^c \) and stable \( W^c \) (non-stable \( W^u \)) manifolds in a non-linear map (17.3) which are tangent to \( E^c \) and \( E^s, E^u \), respectively.

2. If our map is in a canonical form:

\[ \begin{align*}
  x_{t+1} &= \lambda_1 x_t + f(x_t, y_t) \\
  y_{t+1} &= \lambda_2 y_t + g(x_t, y_t)
\end{align*} \] (17.4)
then the center manifold can be written as:

\[ y = h^c(x), \quad h^c(0) = 0; \quad \frac{dh^c}{dx}(0) = 0 \]  

(17.5)

and the flow on the center manifold is:

\[ x_{t+1} = f(x_t, h^c(x_t)) \]  

(17.6)

The manifolds are shown schematically in fig.1.

Figure 17.1: Phase portrait of a linear (a) and corresponding non-linear (b) map with a center manifold

To compute the center manifold for maps we can use exactly the same plan as for ODEs.

### 17.2 Plan for computation of center manifold

**Step1** Put map including non-linear terms into the canonical form (17.1).

**Step2** Find equation for center manifold and add it to the map in the following way

\[
\begin{align*}
    x_{t+1} &= \lambda_1 x_t + f(x_t, y_t) \\
    y_{t+1} &= \lambda_2 y_t + g(x_t, y_t) \\
    y_t &= h^c(x_t)
\end{align*}
\]  

(17.7)

**Step3** Solve system (17.7) using Taylor expansion and find Taylor expansion for \( y = h^c(x) \)

**Step4** Find flow on center manifold as

\[ x_{t+1} = f(x_t, h^c(x_t)) \]

1. Note, that for the system with \( |\lambda_2| = 1, |\lambda_1| \neq 1 \) the equations similar to (17.7) will be:

\[
\begin{align*}
    x_{t+1} &= \lambda_1 x_t + f(x_t, y) \\
    y_{t+1} &= \lambda_2 y_t + g(x_t, y) \\
    x_t &= h^c(y_t)
\end{align*}
\]  

(17.8)
17.3 Map with a parameter

Now let us consider a map with one parameter:

\[
\begin{align*}
    x_{t+1} &= f(x_t, y, c) \\
    y_{t+1} &= g(x_t, y, c)
\end{align*}
\] (17.9)

We can also transform it to the canonical form:

\[
\begin{align*}
    x_{t+1} &= \lambda_1(c)x_t + F(x_t, y, c) \\
    y_{t+1} &= \lambda_2(c)y_t + G(x_t, y, c)
\end{align*}
\] (17.10)

For this map we can also find a center manifold for each parameter value close to \(c = 0\):

\[
y = h^c(x, c)
\] (17.11)

and we can also find a flow on the center manifold:

\[
x_{t+1} = \lambda_1(c)x_t + F(x_t, h^c(x_t, c), c)
\] (17.12)

This flow will contain all the bifurcations of our map. The phase portrait in a two-dimensional phase space can be found by adding the hyperbolic flow along the \(y\) axis, as we did for ODEs.

In such a way we can study all the bifurcations in two variable maps, such as fold, transcritical, pitchfork and flip bifurcation.

The idea of proof, that map (17.10) has the center manifold for each parameter values (17.11), is the same as for ODEs. We use the suspended map which in this case is:

\[
\begin{align*}
    x_{t+1} &= \lambda_1(c_t)x_t + F(x_t, y_t, c_t) \\
    y_{t+1} &= \lambda_2(c_t)y_t + G(x_t, y_t, c_t) \\
    c_{t+1} &= c_t
\end{align*}
\] (17.13)

In case of maps we can also find general conditions for the fold bifurcation similar to those for ODEs. This computation is basically the same as for ODE, so we leave it as an exercise.
Chapter 18

Bifurcations of limit cycles of ODEs

The maps which we have studied in previous chapters arise in different contexts in various systems. One important example of maps arises when we study periodic processes. The periodic behavior of dynamical system usually means that it has a limit cycle. The important questions regarding these limit cycles are to study their stability, their bifurcations and the fate of these limit cycles when we change the parameters of our system.

It turns out that one discrete map plays an important role in studying of the above questions. This map is called the Poincare map.

**Definition 18** The Poincare map, or the first return map $P$ near a periodic orbit $\Gamma$ is defined as

$$x_1 = P(x_0)$$

(i.e., to find $P(x_0)$ we need to follow an orbit which goes through $x_0$ until the next intersection with $L$)

Note, that two Poincare maps defined through the different sections $L$ are topologically equivalent.

Fixed points on Poincare map correspond to limit cycles. Stability of fixed points on Poincare map is equivalent to stability of limit cycles:

**Theorem 16** Let $x_p \subset \Gamma$, where $\Gamma$ is a periodic orbit. Let $P(x)$ is a Poincare map around this orbit, therefore $x_p$ is a fixed point of this Poincare map: $x_p = P(x_p)$. Then $\Gamma$ is stable if $x_p$ is a stable fixed point of $P(x)$ and unstable if $x_p$ is unstable fixed point of $P(x)$.

Formally these stability conditions can be written as:

$$|dP/dx| < 1 \quad stable \quad |dP/dx| > 1 \quad unstable$$

However, in 2D $dP/dx$ cannot be negative. This is because negative $dP/dx$ means that the trajectory which starts inside the limit cycle ($x$ is positive) at the next iteration must go
outside the limit cycle ($x$ should become negative if $dP/dx < 0$). This is impossible as such a trajectory must cross the limit cycle, but the trajectories of the autonomous systems do not cross each other.

Hence in 2D the only possible situations are:

$$dP/dx < 1 \quad \text{stable} \quad dP/dx > 1 \quad \text{unstable}$$

and the only non-hyperbolic situation which can give bifurcations occurs if:

$$dP/dx = 1$$

which in general case gives a fold bifurcation. One case of such bifurcation is drawn below. In this bifurcation we have two limit cycles one of which is stable and the other is non-stable. When we change the parameter value, the non-stable and stable limit cycles move towards each other and disappear.

You can draw similar diagrams for the other cases of fold bifurcation.

Later, we will show that the saddle-node bifurcation of limit cycles can occur close to so-called generalized Hopf bifurcation.

In 3D the Poincare map is two-dimensional. The stability and bifurcations in this case will depend on eigen values of the Jacobian of Poincare map. Here we also have some restrictions on eigen values (the eigen values must have the same sign). However it does not change the situation. And we can have all main 2D bifurcations here:

1. fold(tangent), when one of eigen values equal to 1. Here we get the appearance or disappearance of two limit cycles.

2. flip (period doubling), when one of eigen values equal to -1. Here we get a limit cycle with a double period.

3. Neimark-Sacker, when $\mu = e^{\pm i\phi}$. Here we get an invariant torus, which gives a two-periodic motion.

Study of bifurcations of limit cycles is an important part of study of any dynamical system. In most of the cases it can be done numerically only. Note, that other bifurcations, such as transcritical and pitchfork bifurcation are also possible for the limit cycles. However, they require some special types of symmetry of the Poincare map, which is rare present for 3D ODEs.
Figure 18.2: Saddle-node bifurcation of limit cycles

Figure 18.3: Period doubling (flip) bifurcation of limit cycles

Figure 18.4: Invariant torus after the Neimark-Sacker bifurcation of limit cycles
Chapter 19

Maps on circle

Here we consider maps on a circle ($S_1$) or, in general, maps on an invariant curve. Such systems occur after Neimark-Sacker bifurcation, or in systems under periodic perturbation, or in some other cases.

To write the general expression for such map, let us assume that our invariant curve is close to a circle. Such map should transform one point on this circle to the other point at the same circle. If we use the angle $\phi$ as the coordinate on the circle we can write our map as:

$$\phi_{t+1} = f(\phi_t)$$

One of the important properties of the angle coordinate is that the angles which differ from each other by $2\pi$ are the same. I.e., they represent the same point. Therefore we can always choose the angle between 0 and $2\pi$. Even if our map will formally give us the value $f(\phi_t) = 3.2\pi$ we can write the coordinate of this point as $1.2\pi$, for example. We need such normalization because we want that the same physical points have always the same coordinates.

Mathematically it is written as:

$$\phi_{t+1} = f(\phi_t) \mod 2\pi$$

Now we can re-scale our circle by dividing everything by $2\pi$. We will get the following map:

$$y_{t+1} = f(y_t) \mod 1 \quad (19.1)$$

This map is slightly better than previous because it is really easy to compute $\mod 1$. It is just a fractional part of the right hand side of our map.

So, let us consider a general map on a circle (19.1). Let us also assume that (19.1) is an invertible map. This is some restriction, because the logistic map, for example is not invertible (it can assign the same value of $f(x)$ to two different $x$.) However the Poincare maps of ODEs are invertible, as $P^{-1}(x)$ can be obtained by motion along the orbit in inverse time. For periodic 1D maps invertability basically means that $f(x)$ is a monotonic function of $x$.

As we will see later, it is better to rewrite map (19.1) in the following form:

$$y_{t+1} = y_t + g(y_t) \mod 1 \quad (19.2)$$

where $g(y)$ now accounts for a shift of a point on a circle for one iteration.
19.1 Shift map

The simplest example of such a map is a shift map on a circle:

\[ y_{t+1} = y_t + a \mod 1 \]  

(19.3)

What are fixed points of this map?

\[ y = y + a \mod 1 \quad \text{or} \quad 0 = a \mod 1 \]

What are solutions of this equation? One obvious solution occurs at \( a = 0 \). If \( a = 0 \) then any \( y \) is the fixed point. However what happens if \( a = 1 \)? In this case each iteration of our map is a jump for a whole rotation. And each point \( y_t \) will arrive to the exactly the same point \( y_{t+1} \). Therefore in this case we also have any \( y \) as the fixed point of our map. Obviously it is valid for any \( a = k, k = 0, 1, 2, 3, 4, \ldots \).

Now, if \( a \neq k \). Do we still have interesting orbits on our shift map? Let us consider the case \( a = 1/2 \). In this case any point will jump at each iteration for a half of the circle. It means that after two iterations each point will arrive to the same initial location. i.e., each point here will be a periodic fixed point of the period 2. This is also valid for any \( a = 1/2 + k k = 0, 1, 2, 3, 4, \ldots \). Similarly \( a = 1/3 \) will give us periodic orbits of the period 3, etc. The main results of this study are given by the following theorem:

**Theorem 17**

\( a) \) The shift map (3) has a periodic fixed point of minimal period \( q \), if and only if \( a = p/q \), where \( p \) and \( q \) are integers with no common factors.

\( b) \) The shift map (3) has a non-periodic orbits if and only if \( a \) is an irrational number.

19.2 Non-linear map

As we have seen in our examples the shift map (19.3) has either all the points \( y \subset S_1 \) as fixed points or no fixed points at all. However if we consider a general non-linear map (19.1) then in general we will have a small number of equilibria, given by:

\[ y^* = f(y^*) \mod 1 \]  

(19.4)

Map (19.1) in many respects is similar to usual nonlinear map in 1D. We can use the same criteria for stability: if \( |df/dy| < 1 \) then the equilibrium point will be stable, if \( |df/dy| > 1 \) the point will be not stable. However, because we consider maps on a circle we can get some extra information: If we find one stable fixed point \( y_1 \) then from fig.1 it is reasonable to assume that we also have a non-stable fixed point \( y_2 \). We can study periodic orbits of non-linear map on a circle in a way similar to study of periodic orbits of usual maps. For example for study a period two orbits, we need to consider a double iterated map. For each period two orbit we will get two fixed points on a double iterated map. And because we consider maps on a circle, we also anticipate fixed points of the other stability in between.

We can study fixed points and their stability “locally”, i.e., using partial derivatives their values, etc. However this is an important global characteristic of a map on a circle. It is called the rotation number.
19.3 Rotation number

Now let us introduce an important global characteristic of a map on a circle: rotation number which connects the general non-linear map (2) to the shift map (3).

**Definition 19** The rotation number of the map

\[ y_{t+1} = y_t + g(y_t) \mod 1 \]  \hspace{1cm} (19.5)

is

\[ \rho = \lim_{k\to\infty} \frac{g(y_0) + g(y_1) + \ldots + g(y_{k-1})}{k} \mod 1 \]  \hspace{1cm} (19.6)

There are several theorems regarding the rotation number:

**Theorem 18** If \( g(y) \) is 2 times differentiable function (\( g(y) \subset C^2 \)), then the rotation number \( \rho \) is well defined, i.e., the limit (6) exists and is independent on the initial point \( y_0 \). And,

a) \( \rho \) is rational if and only if \( g(y) \) has a periodic orbit of some period

b) \( \rho \) is irrational if and only if every orbit of \( g \) is dense on \( S_1 \).

**Theorem 19** For the map

\[ y_{t+1} = f(y_t, c) \mod 1 \]  \hspace{1cm} (19.7)

the rotation number \( \rho(f) \) is a continuous function of \( c \).

Note, that for the map (3) the rotation number is \( \rho = a \), if the map has a stable equilibrium as in fig 1a, then the rotation number is \( \rho = 0 \), and if it has a period two orbit (two stable equilibria on a double iterate map) then \( \rho = 1/2 \).

19.4 Phase locking

One of the most interesting phenomena which occur for maps on a circle is a phenomenon of phase locking. This phenomenon means the following: the rotation number of a map instead of smooth changing with change of parameter remains constant in some interval of parameters. Or, in other words, we have an orbit with a period which is locked and does not change when we change the parameters of our system. The reason for phase locking is the following. If we have a period two orbit, for example, than we have a phase portrait as in fig 1b. If we assume
that our equilibria are hyperbolic, then this phase portrait will be the same at small changes of
parameters. Therefore at small variations of parameter we will still have two stable equilibria
on a double iterate map, or the rotation number will be \( \rho = 1/2 \). If we change the parameters
further we can expect that non-stable and stable equilibria disappear via the fold bifurcation.
After that we will loose the phase locking with \( \rho = 1/2 \).

Consider phase locking on the following example:
Consider the standard circular map.
\[
x_{t+1} = \omega + x_t + \frac{\epsilon}{2\pi} \sin 2\pi x_t \mod 1
\]
Let us study in which region we have a phase locking with a rotational number 0. To study it
we need to consider a single iterated map and find its equilibria. Fixed points of this map are:
\[
x_\star = \omega + x_\star + \frac{\epsilon}{2\pi} \sin 2\pi x_\star \mod 1
\]
\[
\sin 2\pi x_\star = -\frac{2\pi \omega}{\epsilon} \text{ if } |\frac{2\pi \omega}{\epsilon}| < 1 \text{ two equilibria}
\]
\[
\text{if } |\frac{2\pi \omega}{\epsilon}| > 1 \text{ no equilibria}
\]
\[
\text{if } |\frac{2\pi \omega}{\epsilon}| = 1 \text{ one equilibrium (} x = 1/4 \text{ or } x = 3/4)\]
So we expect the fold bifurcation in the last case. To check it let us compute a derivative of rhs
at the equilibrium \((x = 1/4 \text{ or } x = 3/4)\):
\[
df/dx = 1 + \epsilon \cos 2\pi x = 1
\]
as \( \cos 2\pi x = 0 \) for \( x = 1/4 \) or \( x = 3/4 \). We can check non-degeneracy conditions for the fold
bifurcation and find that we do have the fold bifurcation here. So we get the following wedge
on the \( \omega, \epsilon \) plane where we have one stable and one instable fixed point.

As we discussed earlier the rotational number of the map in this region will be zero. We
can do the same for a double iterated map and find a region of phase locking with a rotational
number 1/2. It will obviously start at \( \epsilon = 0; \omega = 1/2 \) as at \( \epsilon = 0 \) our map is just a shift map
(3) which at \( \omega = 1/2 \) has a rotation number \( \rho = 1/2 \). Then we can get a similar wedge at the
boundaries of which we have a saddle node bifurcations. The final picture will be the following.
And the rotational number for this map at some \( \epsilon \) will be as shown in fig.19.2b

![Bifurcation diagram of the standard map](image1.png)
![Rotation number graph](image2.png)

Figure 19.2: a- Bifurcation diagram of the standard map. (“Arnold tongues”). b-The graph of
the rotation number as the function of \( \omega \) at fixed \( \epsilon \).

Note that at \( \epsilon = 0 \) the rotation number is a straight line \( \rho = \omega \). So we can clearly see the
effects of non-linearity.

114
19.5 Phase locking and Neimark-Sacker bifurcation

One of the most important situations in which we observe the phenomenon of phase locking is phase locking after the Neimark-Sacker bifurcation. To see it let us consider the dynamics of a map on an invariant curve which occurs after the Neimark-Sacker bifurcation. It is given by:

$$\theta_{t+1} = \theta_t + \phi(\lambda) + NLterms(\lambda)$$  \hspace{1cm} (19.11)

The map (19.11) has some similarities with the standard circular map (19.8) if we use $\phi(\lambda)$ as $\omega$ and the amplitude of $NLterms(\lambda)$ as $\epsilon$. The map in such variables is:

$$\theta_{t+1} = \theta_t + \omega(\lambda) + \epsilon(\lambda)$$  \hspace{1cm} (19.12)

We can assume that this map (19.12) has a phase locking wedges similar to those of the standard circular map. Now, let us consider what will be the dynamics of our system when we change the parameter $\lambda$. For each $\lambda$ we can compute $\omega$ and $\epsilon$, and therefore when we change $\lambda$ we will get a line on the $\omega, \epsilon$ plane. At the point of Neimark-Sacker bifurcation we have “no nonlinear terms”, i.e., it should correspond to $\epsilon = 0$. Later, this line eventually will cross some wedges and we get a phase locking. i.e., period of our orbit will be constant and in some interval independent on the parameter of our system. Further change of parameters will bring our system to the other wedge and to the orbits of other period. In many cases it is possible to find the boundaries of these regions of phase locking numerically by continuation the line of a saddle-node bifurcation for the orbit with the period corresponding to the period of the wedge.

![Figure 19.3: Phase locking after the Neimark-Sacker bifurcation](image-url)
Chapter 20

Homoclinic bifurcation in 2D

Here we consider an important example of a global bifurcation which involves limit cycles. This bifurcation frequently occurs in two dimensional ODEs.

We start with definitions of two important types of limit orbits in 2D: the homoclinic and the heteroclinic orbits:

**Definition 20** The orbit of ODE which approaches an equilibrium point $x_1$ both for $t \to \infty$ and $t \to -\infty$ is called homoclinic to $x_1$.

Note, that in order to have a homoclinic orbit the equilibrium point must have a non-stable and a stable manifolds. Therefore the only equilibrium point in 2D which can have a homoclinic orbit is a saddle point.

**Definition 21** The orbit of ODE which approaches an equilibrium point $x_1$ for $t \to -\infty$ and an equilibrium point $x_2$ for $t \to \infty$ is called heteroclinic to the equilibria $x_1, x_2$.

These orbits are shown in fig.20.1

![Diagram of homoclinic and heteroclinic orbits](image)

**Figure 20.1:** The homoclinic (a) and heteroclinic (b) orbits of ODE

It turns out that systems which have homoclinic or heteroclinic orbits usually are structurally unstable and we can expect bifurcation around the parameter values when the homoclinic or heteroclinic orbits occur.

To study bifurcation which occurs around the homoclinic orbit we will introduce a special function which characterizes how an orbit is close to a homoclinic orbit. For that we consider a non-stable manifold of a saddle point (fig.20.2). If this point has a homoclinic orbit then this
non-stable manifold will arrive to the equilibrium point along the stable manifold (fig.20.2b). If the orbit is not a homoclinic, but close to a homoclinic orbit, then there will be a distance between the non-stable and stable manifolds of the equilibrium(fig.20.2a,20.2c). Let us make a section orthogonal to the stable manifold and measure the distance with a sign between the non-stable and stable manifold. If we denote this distance as $\alpha$ then we will get $\alpha > 0$ in fig.20.2a; $\alpha = 0$ for the homoclinic orbit in fig.20.2b and $\alpha < 0$ in fig20.2c. This $\alpha$ is called a split function. If the transition from fig20.2a to fig.20.2c occurs when we change a parameter of our system, then $\alpha$ will be a function of that parameter.

$$\alpha < 0 \quad \text{a} \quad \alpha = 0 \quad \text{b} \quad \alpha > 0 \quad \text{c}$$

Figure 20.2: The definition of a split function of ODE

Using split function we can formulate theorem about the homoclinic bifurcation for 2D systems.

**Theorem 20** Assume that system (20.1) has an equilibrium $x^*, y^*$ for all small $c$ close to $c^*$. Assume that the equilibrium of system (20.1) is a saddle and $\lambda_1 < 0, \lambda_2 > 0$. Assume that there is an orbit homoclinic to this equilibrium at $c = c^*$.

$$\dot{x} = f(x, y, c)$$
$$\dot{y} = g(x, y, c)$$  \hspace{1cm} (20.1)

Suppose that

1. the trace of the Jacobian at the equilibrium is $\text{tr} J = \lambda_1(c^*) + \lambda_2(c^*) \neq 0$
2. the derivative of the split function $\alpha(c)$ at the bifurcation point $\frac{d\alpha}{dc}(c^*) \neq 0$

Then we have the homoclinic bifurcation with the bifurcation diagrams shown in fig.20.3.

**Proof.** Let us transform our system to a canonical form for linear terms. Then our system can be re-written as:

$$\dot{x} = \lambda_1 x + f(x, y, c) \quad \lambda_1 < 0$$
$$\dot{y} = \lambda_2 y + g(x, y, c) \quad \lambda_2 > 0$$  \hspace{1cm} (20.2)

Therefore the unstable manifold will be along the $y$ axis and the stable manifold along the $x$ axis (fig.20.4).

Consider a section $U_0$ which is close to the equilibrium point $(0,0)$ and is orthogonal to the stable manifold (the dashed line in fig.20.4). Our main idea is to make a map which moves points from the section $U_0$ to the same section along the trajectories of our system. So this map, in some way, is similar to the Poincare map for limit cycles. The thick line in fig.20.4 represents unstable manifold. Therefore the $y$ coordinate of the point of intersection of this line with the section $U_0$ gives us a split function $\alpha$.  

117
In order to find a map which moves points from the section $U_0$ to the same section $U_0$ along the trajectories of our system we use one intermediate step. We will pick up a point $v_0$ on our section $U_0$ and map it first to the point $v_1$ on the line $U_1$ which is also close to the equilibrium. (Line $U_1$ is orthogonal to the unstable manifold.) After that we will map the point $v_1$ to the point $v_2$ on the section $U_0$. The main idea here is the following. Because both sections $U_0$ and $U_1$ are close to the point 0,0 we can use linear approximations of our system and to find the point $v_1$. Then we will find a general form for the map which transforms points from the section $U_1$ to the section $U_0$ and using it we will find the point $v_2$. These two steps together will give us the map of $v_0$ to $v_2$ on the section $U_0$.

Let us find the map $v_0$ to $v_1$. Because the both section $U_0, U_1$ are close to the point 0,0 we
can use linearization of system (20.1). We get the following general solution:

\[
\begin{align*}
\dot{x} &= \lambda_1 x \\
\dot{y} &= \lambda_2 y
\end{align*}
\]

(20.3)

\[
x = C_1 e^{\lambda_1 t} \quad y = C_2 e^{\lambda_2 t}
\]

(20.4)

The trajectory which brings the point \( v_0 \) to \( v_1 \) starts at the point \( x = U_0, y = v_0 \). If we substitute these initial conditions into (20.4) we will get that the following trajectory which is shown in fig.20.4:

\[
x = U_0 e^{\lambda_1 t} \quad y = v_0 e^{\lambda_2 t}
\]

(20.5)

The coordinates of the second point are \( v_1, U_1 \). Therefore we get

\[
y = U_1 = v_0 e^{\lambda_2 T}; \quad T = \frac{1}{\lambda_2} \ln\left(\frac{U_1}{v_0}\right)
\]

therefore

\[
v_1 = U_0 e^{\lambda_1 T} = \frac{U_1}{\lambda_2} e^{\frac{\lambda_1}{\lambda_2} \ln\left(\frac{U_1}{v_0}\right)} = \frac{U_1}{v_0} \frac{\lambda_1}{\lambda_2}
\]

(20.6)

or

\[
v_1 = Av_0 \frac{\lambda_1}{\lambda_2} \quad \text{where} \quad A = U_0(U_1) \frac{\lambda_1}{\lambda_2} > 0
\]

(20.7)

Now the next step is to find a map from \( v_1 \) to \( v_2 \). Let us denote this map as \( \phi(v) \). It has the following properties. Because the point \( v = 0 \) belongs to the unstable manifold, then \( \phi(0) = \alpha \). (fig.20.4). If \( v \neq 0 \) then using Taylor series we get:

\[
\phi(v) = \phi(0) + \phi'(0)v + ...
\]

(20.8)

Here \( \phi(0) = \alpha \). Let us find a sign of the next coefficient \( \phi'(0) \). For that let us assume that \( v > 0 \). The image of this point to the section \( U_0 \) will be above the point \( \alpha \) (fig.20.4), as the trajectory cannot cross the unstable manifold. Therefore \( \phi(v) > \alpha \). This means that \( \phi'(0) > 0 \). If we denote \( \phi'(0) = a \) we can write the following formulae for the function \( \phi \) and therefore for the point \( v_2 = \phi(v_1) \)

\[
v_2 = \alpha + av_1; \quad a > 0
\]

(20.9)

By substituting (20.7) into (20.9) we find the following map from \( U_0 \) to \( U_0 \)

\[
v_2 = \alpha + aU_0 \frac{\lambda_1}{\lambda_2}
\]

(20.10)

Let us study bifurcations of this map. Here we have two different cases: \( trJ = \lambda_1 + \lambda_2 < 0 \) and \( trJ = \lambda_1 + \lambda_2 > 0 \) Because as we assumed \( \lambda_1 < 02 ; \lambda_2 > 0 \) then in the first case we obtain:

\[
-\lambda_1 > \lambda_2 \quad \frac{\lambda_1}{\lambda_2} > 1
\]

similarly in the second case \( trJ = \lambda_1 + \lambda_2 > 0 \) we obtain:

\[
-\frac{\lambda_1}{\lambda_2} < 1
\]

The maps for case 1 and case 2 are presented in fig.20.5. They give the bifurcation diagram presented in fig.20.3.

Note, it was sufficient for our purpose to derive and study these maps for \( v \geq 0 \) only. Note, also, that these maps do not have any sense for case \( v < 0 \). This is because case \( v < 0 \) means that the trajectory in fig.20.4 starts below the stable manifold of the saddle point and therefore will never come to the vicinity of the non-stable manifold located along the positive branch of the \( y \) axis. Therefore it cannot be described using our approach.
Figure 20.5: The maps for various values of $\alpha$ for the case $trJ < 0$ (a), and for the case $trJ > 0$ (b)
Chapter 21

The cusp bifurcation

In this chapter we consider the first bifurcation which occur in systems with two parameters. Let us consider a 1D ODE with two parameters.

\[ \dot{x} = f(x, c, d) \] (21.1)

We know that we can expect the fold (and some other) bifurcations in a system with one parameter. This means that we can find a point at which

\[ f(x^*, c^*) = 0; \quad \text{and} \quad \frac{\partial f}{\partial x}(x^*, c^*) = 0 \] (21.2)

We can expand the right hand side of ODE into the Taylor series around that point and if the second derivative

\[ \frac{\partial^2 f}{\partial x^2}(x^*, c^*) \neq 0, \quad \frac{\partial f}{\partial c}(x^*, c^*) \neq 0 \] (21.3)

we find the fold bifurcation.

In case of system (21.1) we have one additional parameter \(d\). This means that all the derivatives in (21.2) and (21.3) will depend on this parameter. Therefore it is possible that by changing the parameter \(d\) we can find a point where

\[ \frac{\partial^2 f}{\partial x^2}(x^*, c^*, d^*) = 0 \] (21.4)

Note, that this is different from the procedure which we used for the pitch-fork bifurcation. In the case of the pitch-fork bifurcation the derivative (21.3) was also zero, but is was because of a special symmetry of our function. Here we show that we can fulfill condition (21.4) for any generic function \(f\) if it depends on two parameters \(c\) and \(d\).

Let us study bifurcation which can occur around such a point. So let us assume that ODE (21.1) has an equilibrium point at \(x = 0, c = 0, d = 0\) and at this point:

\[ \left\{ \begin{array}{l}
  f(0, 0, 0) = 0 \\
  \frac{\partial f}{\partial x}(0, 0, 0) = 0 \\
  \frac{\partial f}{\partial c}(0, 0, 0) = 0 \\
  \frac{\partial f}{\partial d}(0, 0, 0) = 0
\end{array} \right. \] (21.5)

Then it is possible to prove, that our ODE can be transformed to the following normal form:

\[ \dot{y} = \beta_1 + \beta_2 y + y^3 \] (21.6)
Note, that because our ODE had two parameters, the normal form also has two parameters. This transformation has also some non-degeneracy conditions. They are:

\[
\frac{\partial^3 f}{\partial x^3}(0,0,0) \neq 0 \quad \frac{\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial c} & \frac{\partial f}{\partial d} \\ \frac{\partial^2 f}{\partial x \partial c} & \frac{\partial^2 f}{\partial x \partial d} & \frac{\partial^2 f}{\partial c \partial d} \\ \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial c} & \frac{\partial^2 f}{\partial x \partial d} \end{vmatrix}}{\partial x^3} \neq 0 \quad (21.7)
\]

Now, let us study normal form (21.6). The first question is, how should we represent the results? The idea here is quite simple. We need to compress information. Let us consider a usual fold bifurcation (fig.21.1).

Let us first draw our usual bifurcation diagram. For that let us fix one parameter \(d\) and change the parameter \(c\). Fig21.1a shows usual bifurcation diagram. What is important here? The most important here is the fact that our system can have two different regions of behavior. In region I, it has no equilibria, in region II it has two equilibria one of which is a stable attractor. Where the dynamic of our system changes? At the point where the bifurcation occurs (black circle in fig.). So, let us draw this point on the \(c\) axis. This one point will give us some information about our system. It will tell us that to the left of this point we have no attractors, and to the right of it we will have two equilibria with a certain dynamics.

We were able to represent our diagram on a one-dimensional axis. Now we can do the same for the system with two parameters (fig.21.1b). We use two axes for the parameters. Our point from fig.21.1a will become a point in fig.21.1b. If we now change the second parameter of our system \(d\) the point were the bifurcation occurs move on the \(c, d\) plane and we get a line. This is a new type of lines, so-called the bifurcation line. In our particular case it will show that to the left from this line we will have no equilibria, while to the right from this line we will get two equilibria with one of which being stable.

Now we can formulate our plan. Our plan of study of system with two parameters will be to find all bifurcation lines on the plane of that two parameters, find regions of different behavior and describe the behavior in that regions.

And not let us study the dynamics of the normal form (21.6). Let us consider the "−" case:

\[
\dot{y} = \beta_1 + \beta_2 y - y^3
\]

We need to find bifurcation lines here. The bifurcation which we expect is a fold bifurcation. At bifurcation point we must satisfy two conditions:

\[
f(y^*, c^*, d^*) = 0; \quad \frac{\partial f}{\partial x}(y^* c^* d^*) = 0 \quad (21.8)
\]

Figure 21.1: Fold bifurcation in an ODE with two parameters. (a) Bifurcation diagram for a fixed \(d\); (b) line of fold bifurcation
For our ODE we get:

\[
\begin{align*}
\beta_1 + \beta_2 y - y^3 &= 0 \\
\beta_2 - 3y^2 &= 0
\end{align*}
\]  

(21.9)

or

\[
\begin{align*}
y^2 &= \frac{\beta_2}{3} \\
\beta_1 &= y^3 - \beta_2 y = -2y^3 \\
y^3 &= -\frac{\beta_1}{2} \\
\left(\frac{\beta_2}{3}\right)^3 &= (-\frac{\beta_1}{2})^2 \\
4(\beta_2)^3 &= 27(\beta_1)^2
\end{align*}
\]

(21.10)

So the equation for the line of the fold bifurcation is:

\[
\beta_2 = 3\sqrt[3]{\frac{27}{4}\beta_1^2}
\]

(21.11)

The graph of this line is shown in fig.21.2.

![Diagram for the cusp bifurcation](image)

Figure 21.2: Diagram for the cusp bifurcation

Here we see that our system has two regions of different behavior. Let us find a phase portraits in that two regions.

Consider region 2. We expect that the behavior in all points form region 2 will be similar. Let us consider, for example, a point $\beta_2 = 0, \beta_1 = 1$ is inside this region. Our ODE for that parameter values is:

\[
\dot{y} = 1 - y^3
\]

(21.12)

It has a phase portrait shown in fig.21.3b. We see that we have one stable equilibrium.

Now consider a point from region 1. For example a point $\beta_2 = 1, \beta_1 = 0$. The ODE becomes:

\[
\dot{y} = y - y^3 = y(1 - y^2) = y(1 - y)(1 + y)
\]

(21.13)

The phase portrait is show in fig.21.3a. Here we have three equilibria. Two of them are stable and one non-stable.

**Conclusion.** If we continue a line of the fold bifurcation and find a point of the cusp bifurcation, i.e., the point where $\frac{d^2 f}{dc^2} = 0$, then we expect that our line of the fold bifurcation will suddenly change its direction at this point. Around this cusp point we expect two regions of different behavior involving at least three equilibria of our system.
Figure 21.3: Phase portraits for the cusp bifurcation. (a) for region 1; (b) for region (2).
Chapter 22

Generalized Hopf bifurcation

In this chapter we consider another bifurcation: the generalized Hopf bifurcation.

If we consider a system with one parameter:

\[
\begin{align*}
\dot{x} &= f(x, y, c) \\
\dot{y} &= g(x, y, c)
\end{align*}
\] (22.1)

then we can have the Hopf bifurcation if the eigen values of the Jacobian matrix at the equilibrium point are:

\[\lambda = \alpha(c) \pm i\beta(c); \quad \alpha(0) = 0\] (22.2)

and the following non-degeneracy conditions are valid:

\[
\frac{\partial \alpha}{\partial c}(0) \neq 0; \quad \text{and} \quad \beta(0) \neq 0; \quad \text{and} \quad I \neq 0; \quad (22.3)
\]

where \(I\) is the stability index.

For system with two parameters we can find a point where one of the non-degeneracy conditions does not hold. The generalized Hopf bifurcation occurs when \(I = 0\).

In more details. Assume that we have a system with two parameters:

\[
\begin{align*}
\dot{x} &= f(x, y, c, d) \\
\dot{y} &= g(x, y, c, d)
\end{align*}
\] (22.4)

Assume that there is equilibrium of this system where the following conditions for the eigen values of the Jacobian matrix hold:

\[\lambda = \alpha(c*, d*) \pm i\beta(c*, d*); \quad \alpha(c*, d*) = 0\] (22.5)

and the stability index:

\[I(c*, d*) = 0\] (22.6)

It is possible to prove that in a generic case such system (22.4) can be transformed to the following normal form:

\[
\begin{align*}
\dot{r} &= r(\beta_1 + \beta_2 r^2 \pm r^4) \\
\dot{\theta} &= 1
\end{align*}
\] (22.7)

where \(r, \phi\) are polar coordinates. The sign \(\pm\) in this equation is determined by the stability index, which is similar to the stability index for the Hopf bifurcation. This stability index includes various derivatives of the right hand side function, and can be found in special books.
on bifurcation theory. Because we will not study analytically systems with generalized Hopf bifurcation, we do not present the expression for this stability index, which is quite complex.

Let us study normal form (22.7) for the \( \beta'' - \beta' \) case:

\[
\begin{align*}
\dot{r} &= r(\beta_1 + \beta_2 r^2 - r^4) \\
\dot{\theta} &= 1
\end{align*}
\]  

(22.8)

Figure 22.1: a: Graph of function \( \beta_1 + \beta_2 x - x^2 \) for various \( \beta_1 < 0, \beta_2 < 0 \). b: The direction of parameter change. Analysis of phase portraits is presented in fig.22.2

System (22.8) has two independent equations: one for the variable \( r \), and one for the variable \( \theta \). Equation for the variable \( \theta \), which is \( \dot{\theta} = 1 \) gives just rotation of a trajectory, while the equation for the variable \( r \) gives some 'real dynamics'. Let us find possible types of dynamics of the equation for \( r \) for different values of \( \beta_1, \beta_2 \). The equation for \( r \) is:

\[
\dot{r} = r(\beta_1 + \beta_2 r^2 - r^4)
\]  

(22.9)

Equation (22.9) always has one equilibrium at \( r = 0 \). In order to find other equilibria let us consider graphs of the function \( y = (\beta_1 + \beta_2 r^2 - r^4) \) for various values of \( \beta_1, \beta_2 \). Let us denote \( r^2 = x \), we get:

\[
y = \beta_1 + \beta_2 x - x^2
\]  

(22.10)

Note, that because \( x = r^2 \), we are interested only in positive roots of equation (22.10). Let us find them graphically (fig.22.1). Graph of the function \( y = \beta_1 + \beta_2 x - x^2 \) is a parabola opened to below. Parameter \( \beta_1 \) gives the point of intersection of the parabola with the \( y \) axis (at \( x = 0 \)), and changing of \( \beta_1 \) shifts parabola in the \( y \) direction. Parameter \( \beta_2 \) gives a slope of the tangent line to this parabola at point \( x = 0 \). We see (fig.22.1) that if \( \beta_1 < 0, \beta_2 < 0 \), function \( y = \beta_1 + \beta_2 x - x^2 \) does not have any roots for non-negative \( x \). As we will see further, this means that at these values of parameters the phase portrait of our system is quite simple. Let us find this phase portrait and its modifications if we change parameter \( \beta_1 \) from a negative to a positive value (fig.22.1b)

For that we need to draw graphs of function \( y = r(\beta_1 + \beta_2 r^2 - r^4) \) for negative \( \beta_2 \) and for \( \beta_1 \) increasing from some negative to some positive value. Because increase of \( \beta_1 \) just shifts the graph from fig.22.1a upwards, graphs of \( y = \beta_1 + \beta_2 x - x^2 \) will be as shown at the upper row in fig.22.2. Function \( y = r(\beta_1 + \beta_2 r^2 - r^4) \) is just \( y = \beta_1 + \beta_2 x - x^2 \) multiplied by
a non-negative number $r$. This means that is graph of $y = \beta_1 + \beta_2 x - x^2$ is positive, graph of $y = r(\beta_1 + \beta_2 r^2 - r^4)$ is also positive; if graph of $y = \beta_1 + \beta_2 x - x^2$ is negative, graph of $y = r(\beta_1 + \beta_2 r^2 - r^4)$ is also negative. The only difference will be at $r = 0$, because here the function $y = r(\beta_1 + \beta_2 r^2 - r^4)$ will be always zero.

Using these properties we can easily draw graphs of function $y = r(\beta_1 + \beta_2 r^2 - r^4)$ for the negative, zero and positive values of $\beta_1$ (fig. 22.2 middle row) and phase portraits of equation $\dot{r} = r(\beta_1 + \beta_2 r^2 - r^4)$ (the lower row).

Figure 22.3: Scheme of the generalized Hopf bifurcation.
Fig. 22.2 (lower row) shows that at $\beta_1 < 0$, equation (22.9) has a stable equilibrium at $r = 0$, which means that if we add the rotation along the theta discussed above, will get a stable spiral in the complete system (22.8). The phase portrait for this case is shown in fig.22.7), case 1. When $\beta_1$ becomes positive, $r = 0$ equilibrium becomes unstable and we get a stable equilibrium at some non-zero $r$. As we discussed earlier, non-zero equilibrium means a limit cycle. Therefore, in region 2 we have a stable limit cycle and a non-stable spiral at $r = 0$. We can interpret these changes as a supercritical Hopf bifurcation which occurs at $\beta_2 = 0$. The phase portrait for this case is shown in fig.22.7), case 2,2’. We can also draw a line of supercritical Hopf bifurcation at $\beta_1 = 0$, $\beta_2 > 0$. Because the stability index of this Hopf bifurcation is negative, we call this line as $H^-$ on fig.22.8.

Figure 22.4: Graphs of the function $y = \beta_1 + \beta_2 x - x^2$ (the left picture) and $y = r(\beta_1 + \beta_2 r^2 - r^4)$ (the right picture) for various values of $\beta_2$, at $\beta_1 > 0$.

Now let us change the parameters from region 2 to region 2’ in fig.22.3. At such parameter changes $\beta_1$ is constant and $\beta_2$ increases from some negative to some positive value. For function $y = \beta_1 + \beta_2 x - x^2$ this means that parabola always starts at the same point $y = \beta_1, x = 0$, but its slope at this point increases from some positive to some negative value. We see from fig.22.4 that such modifications do not change number of zeros of function $y = \beta_1 + \beta_2 x - x^2$ and therefore of function $y = r(\beta_1 + \beta_2 r^2 - r^4)$ and the shift to region 2’ does not cause any bifurcations: we still have a non-stable equilibrium at $r = 0$ and a stable equilibrium (stable limit cycle) at $r > 0$.

Now we proceed from region 2’ to region 3. At this interval $\beta_2$ is has a constant positive value and $\beta_1$ decreases from some positive to some negative value. This means that graph of function $y = \beta_1 + \beta_2 x - x^2$ shifts below, and one extra zero occurs at $\beta_1 = 0$ (fig.22.5). Therefore for function $r(\beta_1 + \beta_2 r^2 - r^4)$ number of zeros changes from two to three which results in changes of phase portrait shown in fig.22.6. We see, that when $\beta_1$ crosses from positive to negative values the equilibrium $r = 0$ becomes stable and we get an extra non-stable equilibrium (non-stable limit cycle). This can be interpreted as a sub-critical Hopf bifurcation, and the phase portrait will be as as in fig.22.7 region 3. We can also draw a line of sub-critical Hopf bifurcation at $\beta_1 = 0, \beta_2 > 0$. Because the stability index of this Hopf bifurcation is positive, we call this line as $H^+$ on fig.22.8.

Now we return back to region 1.

We see that decreasing of $\beta_2$ (slope of $y = \beta_1 + \beta_2 x - x^2$ at $x = 0$) moves closer two non-zero equilibria fig.22.6 and at some value $\beta_{crit}$ these two equilibria disappear via a saddle-node bifurcation. Because non-zero equilibria correspond to limit cycles, this bifurcation will be a saddle-node bifurcation of limit cycles. The line of a saddle-node bifurcation for limit cycles must be located in region 3.
To find the line of the saddle-node bifurcation for limit cycles let us find non-trivial equi-libria of our equation (22.10)

\[ \dot{r} = r(\beta_1 + \beta_2 r^2 - r^4) \] (22.11)

For non-zero equilibria we get

\[ \beta_1 + \beta_2 r^2 - r^4 = 0 \]
\[ r^4 - \beta_2 r^2 - \beta_1 = 0 \] (22.12)

\[ r_{1,2}^2 = \frac{\beta_2 \pm \sqrt{\beta_2^2 + 4\beta_1}}{2} \]

The transition from the case when we have two roots to the case when we have zero roots means a saddle-node bifurcation for limit cycles. This occurs when the discriminant of our equation becomes negative at the point:

\[ \beta_2 = \pm \sqrt{-4\beta_1} \] (22.13)

and the root of (22.12) is:

\[ r_{1,2}^2 = \frac{\beta_2}{2} \] (22.14)

Equation (22.13) gives, in principle, the line of a saddle-node bifurcation for limit cycles. However not all the branches of (22.13) correspond to the real bifurcation. This is because in
Figure 22.6: Graphs of the function \( y = \beta_1 + \beta_2 x - x^2 \) and \( y = r(\beta_1 + \beta_2 r^2 - r^4) \) for various values of \( \beta_2 \), at \( \beta_1 < 0 \). The lower panel shows a phase portrait of equation \( \dot{r} = r(\beta_1 + \beta_2 r^2 - r^4) \).

In our case \( r_{1,2}^2 \) must be positive. It follows from equation (22.14) that \( r_{1,2}^2 \) is positive only for positive \( \beta_2 \). Therefore the only line of saddle-node bifurcation which we get here is given by:

\[
\beta_2 = +\sqrt{-4\beta_1} \quad (22.15)
\]

This line is show as he dotted line in fig.22.8. The phase portraits for the generalized Hopf bifurcation are given in fig.22.7.

**Conclusion.** If we found a point of the generalized Hopf bifurcation then there are three lines of bifurcation which start from that point: the line of sub-critical Hopf bifurcation, the line of supercritical Hopf bifurcation and the line of saddle-node bifurcation of limit cycles.
Figure 22.7: Phase portraits around the generalized Hopf bifurcation. The numbers 1,2,2’,3 correspond to the values of parameters shown in fig.22.3

Figure 22.8: Generalized Hopf bifurcation.
Chapter 23

Bogdanov-Takens bifurcation

In this chapter we consider another bifurcation: the Bogdanov-Takens bifurcation. This bifurcation occurs in systems with two parameters

\[
\begin{align*}
\dot{x} &= f(x, y, c, d) \\
\dot{y} &= g(x, y, c, d)
\end{align*}
\]  

(23.1)

around the equilibrium point where both eigen values of the Jacobian matrix are zero:

\[\lambda_1(c^*, d^*) = 0; \quad \lambda_2(c^*, d^*) = 0\]  

(23.2)

The first question is: how can we arrive to this point when we change the parameter values. One way of getting such a point is to continue a line of the fold bifurcation on the \(c, d\) plane. We know that at the line of the fold bifurcation one of the eigen values of (23.1) is zero (e.g. \(\lambda_1 = 0\)). It is reasonable to assume, that the other eigen value \(\lambda_2\) can become zero at some point on this line. And if we will continue this line further \(\lambda_2\) will change its sign. Therefore the Bogdanov-Takens point can occur on the line of the fold bifurcation on the \(c, d\) plane (fig.23.1a). Note, that the condition \(\lambda_1 = 0\) can potentially give us other bifurcations, such as transcritical, or pitchfork bifurcation. However, both the transcritical and the pitchfork bifurcations require some extra conditions, i.e., for the pitchfork we need our function to be symmetric, etc. We do not study such special cases here, and assume that \(\lambda_1 = 0\) means a fold bifurcation in system (23.1).

However, there is another possibility for arriving to the Bogdanov-Takens point. We can continue a line of the Hopf bifurcation. As we know on the line of the Hopf bifurcation the eigen values are complex and the real part of the eigen values is zero (\(\lambda = \pm i\beta\)). One of the non-degeneracy conditions for the Hopf bifurcation is \(\beta \neq 0\). However, it can happen that at some point on the Hopf line that \(\beta = 0\). This usually means that at further parameter change eigen values become real. However, at the point where \(\beta = 0\) the eigen values \(\lambda_{1,2} = \pm i0 = 0\), i.e., we also have the Bogdanov-Takens point. Therefore our conclusion is that we can arrive to the Bogdanov-Takens point along the Hopf line and this line will end at this point (fig.23.1b).

It turns out that our preliminary analysis is right and we do find the lines of the fold and the Hopf bifurcation around the Bogdanov-Takens point. But we also find some extra information.

The idea of study of this bifurcation is the same as for other bifurcations. We need to find a normal form and study it. It is possible to prove that any generic two parametric system (23.1) near the Bogdanov-Takens point (23.2) can be transformed into the following normal form:

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= \beta_1 + \beta_2 y + x^2 \pm xy
\end{align*}
\]  

(23.3)
Figure 23.1: Scheme of the Bogdanov-Takens bifurcation.

Note, that the sign ± is determined by the sign of several coefficients of Maclaurin series expansion of the right hand side of system (23.1).

Let us study system (23.3) for the "+" case:

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= \beta_1 + \beta_2 y + x^2 + xy
\end{align*}
\]  

(23.4)

1. Equilibria points.

\[
\begin{align*}
y &= 0 \\
\beta_1 + \beta_2 y + x^2 + xy &= 0 \\
\beta_1 + x^2 &= 0
\end{align*}
\]  

(23.5)

Therefore at \( \beta_1 < 0 \) we have two equilibria:

\[
x_1 = \sqrt{-\beta_1} \quad y_1 = 0 \quad \text{and} \quad x_2 = -\sqrt{-\beta_1} \quad y_2 = 0 \quad \beta_1 < 0
\]

\( \text{no equilibria if} \ \beta_1 > 0 \)  

(23.6)

2. Jacobian. The Jacobian matrix for system (23.3) at equilibria \((\pm \sqrt{-\beta_1}, 0)\) is:

\[
J = \begin{pmatrix}
0 & 1 \\
2x + y & \beta_2 + x
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
\pm 2\sqrt{-\beta_1} & \beta_2 \pm \frac{1}{\sqrt{-\beta_1}}
\end{pmatrix}
\]  

(23.7)

3. Fold bifurcation.

We know that the necessary condition for the fold bifurcation is \( \det J = 0 \) which gives us:

\[
\det J = \mp 2\sqrt{-\beta_1} = 0; \quad \text{or} \quad \beta_1 = 0
\]  

(23.8)

The line of the fold bifurcation is shown as F line in fig.23.2a.
4. Equilibria types. In turns out that for the further analysis we need to study types of equilibria which we get via the fold bifurcation. As we know from eq (23.6) both equilibria are located on the $x$ axis and occur at negative values of $\beta_1$. However, it turns out, that equilibria types depend on parameter $\beta_2$. Let us schematically denote in fig.23.2a two regions: 3 at positive $\beta_2$ and 2 at negative $\beta_2$. For simplicity we assume that we study equilibria types just after the bifurcation, i.e., $\beta_1$ is a negative number which is slightly less than zero. Let find equilibria type using the $\det J$ test. We know from eq (23.8) that $\det J = \pm 2\sqrt{-\beta_1}$, where the $'-'$ sign corresponds to the first equilibrium $x_1 = \sqrt{-\beta_1}$, $y_1 = 0$ and the $'+'$ sign corresponds to the second equilibrium $x_2 = -\sqrt{-\beta_1}$, $y_2 = 0$. Therefore at equilibrium $(x_1, y_1)$ $\det J$ is always negative, thus this equilibrium is always a saddle point. To find type of the second equilibrium we need to compute the trace of the Jacobian.

$$trJ = \beta_2 + x_2 = \beta_2 - \sqrt{-\beta_1}$$

Because we assumed that $\beta_1$ is a small number compared to $\beta_2$ we conclude that $trJ \approx \beta_2$. This means that for $\beta_2 > 0$, i.e., in region 3, $trJ > 0$; for $\beta_2 < 0$, i.e., in region 2, $trJ < 0$. The discriminant

$$D = (trJ)^2 - 4\det J \approx \beta_2^2 \pm 8\sqrt{-\beta_1} \approx \beta_2^2 > 0$$

i.e., $D$ is positive and we have real roots. Note, that this result is valid for small $\beta_1$ only, i.e., when we can apply the approximate formulae. Our conclusion is: equilibrium $x_2, y_2$ is a stable node in region 2 ($\beta_2 < 0$) and a non-stable node in region 3 ($\beta_2 > 0$). We represent these equilibria graphically in fig.23.2bc. Note, that equilibria after the fold bifurcation are always connected to each other via the central manifold. Because stable node has the incoming trajectories only, it will be connected to the non-stable manifold of the saddle, while non-stable node will be connected to the stable manifold of the saddle.

We see, that the phase portraits in regions 2 and 3 are quite different. This means that if we move from region 3 to region 2 (decrease parameter $\beta_2$ from some positive to some negative value) we need to have: (1) transformation of a non-stable node to a stable node; (2) global change of interconnections between the saddle and the node. It turns out that such changes occur via a quite complex sequence of bifurcations. We can get an idea about one of these bifurcations from fig.23.3, which is similar to fig.6.6. We
see that in order to make a smooth transition from a non-stable to a stable node on the det-tr plane, the non-stable node first becomes a non-stable spiral, then it crosses the line det$J = 0$ and becomes a stable spiral and then it becomes a stable node. As we know, the transformation of non-stable spiral to a stable spiral via the point det$J = 0$ means the Hopf bifurcation. Thus we expect the line of Hopf bifurcation between regions 3 and 2. The bifurcation which accounts for transformation of interconnections between the saddle and the node in fig.23.2 will be studied later.

Figure 23.3: Transformation of non-stable to stable node on the det tr plane

5. Hopf bifurcation. Let us find the line of the Hopf bifurcation. The necessary conditions for the Hopf bifurcation are: $trJ = 0, detJ > 0$.

In our case $detJ = \mp 2\sqrt{-\beta_1}$ (see (23.8)). Therefore $detJ$ is positive only at the negative equilibrium $(x_2 = -\sqrt{-\beta_1}, y_2 = 0)$. Its value at that point is $detJ = \sqrt{-\beta_1}$. Let us find when $trJ = 0$ at this point $(-\sqrt{-\beta_1}, 0)$. This expression will give us the line of the Hopf bifurcation on the $\beta_1, \beta_2$ plane:

$$trJ = \beta_2 - \sqrt{-\beta_1} = 0; \quad \beta_2 = \sqrt{-\beta_1} \quad \beta_1 < 0 \quad (23.9)$$

This line of the Hopf bifurcation is shown in fig.23.4.

Now let us find type of the Hopf bifurcation. For this we need to study our system at the bifurcation point and compute a stability index as described in section 7.4. The Jacobian of our system at the bifurcation value of parameters and at equilibrium $x_2, y_2$ is given by

$$J = \begin{pmatrix} 0 & 1 \\ -2\sqrt{-\beta_1} & \beta_2 - \sqrt{-\beta_1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2\sqrt{-\beta_1} & 0 \end{pmatrix} \quad (23.10)$$

here we used the fact that at the bifurcation point $\beta_2 = \sqrt{-\beta_1}$.

We see that in general, our system is not in the canonical form and we cannot apply directly the test from section 7.4. However for one value of the parameter $2\sqrt{-\beta_1} = 1$ the Jacobian will be in the canonical form: $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. 

135
Let us find type of Hopf bifurcation for this value of the parameter and assume that for other values the result does not change. For that let us put $\beta_1 = -\frac{1}{4}$. Due to condition of the Hopf bifurcation (23.9) $\beta_2 = \frac{1}{2}$ and our system (23.4) at the bifurcation point becomes:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{1}{4} + \frac{1}{2}y + x^2 + xy \end{cases} \quad (23.11)$$

Now let us apply the stability index (7.50). To put our system to form (7.50) we need to shift equilibrium point $(x = -\frac{1}{2}, y = 0)$ to zero. It is obvious that we can do it by the shift of the variables $x_{new} = x_{old} + \frac{1}{2}$, $y_{new} = y_{old}$. In the new variables our system becomes:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x + x^2 + xy \end{cases} \quad (23.12)$$

Thus our system is transformed into the form (7.50) with $\omega = 1$, $Y^1 = 0$, $Y^2 = x^2 + xy$. It is easy to find that $I = 2$. This means that equilibrium at the bifurcation point is unstable and we have a sub-critical Hopf bifurcation here (occurrence of a non-stable limit cycle).

Now let us draw the phase portraits. In region 1 we have no equilibria and the phase portraits are just parallel trajectories fig23.5.1. To draw the exact phase portraits in regions 2 and 3 we need to compute eigen values and eigen vectors at the equilibria. However from fig.23.2 we know how they look schematically close to the axis $\beta_2$. We also know that if we move between regions 3 and 2 the nodes will be transformed into the spirals. Note, that spiral and node are topologically equivalent and transition between them is not a bifurcation. Therefore, for simplicity let us draw the phase portraits in regions 2 and 3 for spirals as shown in fig23.5.2 and fig23.5.3. The equilibrium at positive $x$ is always a saddle point.
Now we need to decide what happens at the Hopf bifurcation line. As we know we have a sub-critical Hopf bifurcation and therefore we have a formation of a non-stable limit cycle. We also know that after the Hopf bifurcation a non-stable limit cycle has a stable spiral inside it. Therefore, if we now go from region 3 to region 2 as shown in fig.23.2b we will cross the Hopf line at some point. After the Hopf bifurcation the spiral from fig.23.5.3 becomes stable and we get a non-stable limit cycle. Therefore after the Hopf bifurcation the phase portrait will be as in fig.23.5.4.

If we continue our motion further we will arrive to region 2. However we see that the phase portraits fig.23.5.4 and fig.23.5.2 are different, therefore we must have some bifurcation after the Hopf bifurcation. This also means that we must have some bifurcation line after the Hopf line. To find which bifurcation do we have here let us compare the phase portraits in fig.23.5.4 and fig.23.5.2. We showed the location of non-stable and stable manifolds for that phase portraits in fig.23.6. We see that in region 4 the stable manifold is between the stable spiral and unstable manifold, while in region 2 the stable spiral is connected to the non-stable manifold. Therefore there exists a point between regions 2 and 4 where non-stable manifold crosses the stable manifold, and we have a homoclinic orbit fig.23.6h. After this homoclinic orbit non-stable limit cycle disappears via the homoclinic bifurcation and we get the phase portrait fig.23.5.2.

![Figure 23.6: The location of manifolds around the homoclinic bifurcation. Numbers 2,4 correspond to the numbers from fig.23.5. h shows the homoclinic orbit.](image)

The line on homoclinic bifurcation cannot be computed analytically. This line (line h) is schematically presented in fig.23.7 together with other bifurcation lines. The corresponding phase portraits are as in fig.23.5.

![Figure 23.7: The bifurcation lines in the Bogdanov-Takens bifurcation.](image)

**Conclusion.** If we found a point of the Bogdanov-Takens bifurcation then there are three
lines of bifurcations which start from that point: the line of the Hopf bifurcation, the line of the
fold bifurcation and the line of the homoclinic bifurcation.

Note, that in numerical studied the line of homoclinic bifurcation can be studied as the line
where the period of limit cycle goes to infinity. In locbif it can be done by continuation of a
limit cycle of a sufficiently long period along the parameters of our system.
Chapter 24

PDE models in biology

24.1 Gradient

Assume that we have a function of two variables \( f(x, y) \). We can find derivatives of this function in the \( x \) direction \( \left( \frac{\partial f}{\partial x} \right) \) and in the \( y \)-direction \( \left( \frac{\partial f}{\partial y} \right) \). Now let us find a derivative of this function in an arbitrary direction \( l \) (fig. 24.1a).

![Figure 24.1: a-the directional derivative; b-the gradient of the function \( f(x, y) \).]

The value of \( f \) at point \( l \) with the coordinates \( (x + dx, y + dy) \) is given by

\[
f(x + dx, y + dy) = f(x, y) + \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy
\]  

(24.1)

The length of the vector \( l \) is

\[
l = \sqrt{dx^2 + dy^2}
\]  

(24.2)

Therefore the derivative is:

\[
\frac{df}{dl} = \frac{f(x + dx, y + dy) - f(x, y)}{l} = \frac{\partial f}{\partial x} \frac{dx}{l} + \frac{\partial f}{\partial y} \frac{dy}{l}
\]  

(24.3)

Because \( \frac{\dot{x}}{l} = \cos \phi \) and \( \frac{\dot{y}}{l} = \sin \phi \) we get:

\[
\frac{df}{dl} = \frac{\partial f}{\partial x} \cos \phi + \frac{\partial f}{\partial y} \sin \phi
\]  

(24.4)
Now, let us re-write (24.4) in another form which we will use later. For that let us formally introduce a vector $\nabla f$ which has the $x$-coordinate $\frac{\partial f}{\partial x}$ and the $y$-coordinate $\frac{\partial f}{\partial y}$ (fig.24.1b). This vector is called the gradient of the function $f$. Using the gradient we can re-write

\[
\frac{\partial f}{\partial x} = |\nabla f| \cos \gamma; \quad \frac{\partial f}{\partial y} = |\nabla f| \sin \gamma \quad \text{where} \quad |\nabla f| = \sqrt{\frac{\partial f^2}{\partial x} + \frac{\partial f^2}{\partial y}} \tag{24.5}
\]

using the gradient we can re-write the directional derivative (24.4) as:

\[
\frac{df}{dl} = |\nabla f| \cos \gamma \cos \phi + |\nabla f| \sin \gamma \sin \phi = |\nabla f| \cos (\gamma - \phi) \tag{24.6}
\]

Now we can answer to the following two questions.

What is the maximal value of the directional derivative of the function? It is $|\nabla f|$.

In which direction is the derivative of the function maximal? The derivative is maximal when $\cos (\gamma - \phi) = 1$, or when $\gamma = \phi$. i.e., the derivative is maximal in the direction which coincides with the direction of the gradient of the function.

So the gradient is an important vector for our system. It shows the direction of the steepest slope of our function.

The second important property of the gradient is that if we consider the lines which are given by the equations

\[f(x, y) = c\]

then the gradient vector at a point $x^*, y^*$ is perpendicular to the level line which goes through the point $x^*, y^*$.

### 24.2 Main PDEs

#### 24.2.1 conservation equation

The conservation equation describes the general changes of spatial distributions in a system. Assume that we have a one dimensional system in which we have a substance with a concentration $c$. Let us derive how this concentration will change in time and space. Let us consider a small part of our system from the points $x$ to $x + dx$ (fig.24.2). The total number of particles in this cylinder is

\[N = cA dx \quad \tag{24.7}\]

where $c$ is the concentration of particles, $A$ is the area of the cross section of the cylinder and $dx$ is the length of the small cylinder in fig.24.2. The rate of change of particles in our cylinder can be caused by entry of new particles through the section $x$, minus departure of particles through the section $x + dx$, plus rate of the local degradation/production of particles in our small cylinder.

![Figure 24.2:](image-url)
\[
\left( \text{rate of change of particles in } x, x+dx \right) = \left( \text{rate of entry into } x, x+dx \right) - \left( \text{rate of departure from } x, x+dx \right) \pm \left( \text{rate of local degradation or creation per unit time in } x, x+dx \right)
\]

(24.8)

The same relationship can be expressed in the following way in terms of partial derivatives:

\[
\frac{\partial (cAdx)}{\partial t} = AJ(x) - AJ(x + dx) \pm \sigma Adx
\]

(24.9)

here \( J(x) \) is a flux of particles at the section \( x \) which means number of particles crossing a unit area at \( x \) in the \( x \) direction per unit of time.

If we cancel \( A \) in the both sides of (24.9) and divide each term by \( dx \) we get:

\[
\frac{\partial c}{\partial t} = \frac{J(x) - J(x + dx)}{dx} \pm \sigma
\]

(24.10)

If \( dx \) goes to zero the term \( \frac{J(x) - J(x + dx)}{dx} \) becomes \( -\frac{\partial J}{\partial x} \) and we get the following conservation equation:

\[
\frac{\partial c}{\partial t} = -\frac{\partial J}{\partial x} \pm \sigma
\]

(24.11)

If we do the same in a multi-dimensional space then we get the same expressions for the terms \( \frac{\partial c}{\partial t} \) and \( \pm \sigma \). However the flux term will be changed. This is because the flux in a 2D and 3D media is a vector \( \vec{J} \) as the flux depends on orientation of a surface. The derivation of the conservation equation in 2D gives the following result:

\[
\frac{\partial c}{\partial t} = -\frac{\partial J_x}{\partial x} - \frac{\partial J_y}{\partial y} \pm \sigma
\]

(24.12)

where \( J_x, J_y \) are the \( x \)- and \( y \)-components of the flux vector \( \vec{J} \).

In order to model real biological systems we need to write expressions for the flux \( J \) and for the production term \( \sigma \).

### 24.2.2 convection

If particles move with a given velocity \( \vec{v} \) then the flux vector is:

\[
\vec{J} = c\vec{v}
\]

(24.13)

and the equation describing the convection of particles in 1D medium in absence of production/degradation term is:

\[
\frac{\partial c}{\partial t} = -\frac{\partial (cv)}{\partial x} = -v \frac{\partial c}{\partial x} - c \frac{\partial v}{\partial x}
\]

(24.14)

and in 2D

\[
\frac{\partial c}{\partial t} = -\frac{\partial (cv_x)}{\partial x} - \frac{\partial (cv_y)}{\partial y} = -v_x \frac{\partial c}{\partial x} - c \frac{\partial v_x}{\partial x} - v_y \frac{\partial c}{\partial y} - c \frac{\partial v_y}{\partial y}
\]

(24.15)
24.2.3 chemotaxis

Let us describe motion of bacteria attracted by some chemical. If the concentration of bacteria is \( c \) and the concentration of the attractant is \( b \) then it is known that bacteria move in the direction of the gradient of the attractant \( \nabla b \) and the velocity is proportional to the value of this gradient:

\[
\vec{v}_{\text{bacteria}} = \mu \nabla b
\]  

(24.16)

where \( \mu \) is the motility coefficient.

If we now use equation (24.13) for the flux we get:

\[
\vec{J} = c \vec{v} = c \mu \nabla b
\]  

(24.17)

Therefore we get the following equation for chemotaxis bacteria in absence of the birth/death term.

In 1D

\[
\frac{\partial c}{\partial t} = -\frac{\partial (c \mu \frac{\partial b}{\partial x})}{\partial x} = -\mu \frac{\partial b}{\partial x} \frac{\partial c}{\partial x} - c \mu \frac{\partial^2 b}{\partial x^2}
\]  

(24.18)

or in 2D

\[
\frac{\partial c}{\partial t} = -\frac{\partial (c \mu \frac{\partial b}{\partial x})}{\partial x} - \frac{\partial (c \mu \frac{\partial b}{\partial y})}{\partial y} = -\mu \frac{\partial b}{\partial x} \frac{\partial c}{\partial x} - \mu \frac{\partial b}{\partial y} \frac{\partial c}{\partial y} - c \mu \left( \frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial y^2} \right)
\]  

(24.19)

24.2.4 the diffusion equation

If we have a process of diffusion of a substance, then it is possible to show that the flux is directly proportional to the concentration gradient and directed to the side which is opposite to the gradient. Therefore we can write

\[
\vec{J} = -D \nabla c
\]  

(24.20)

where \( D \) is the proportionality coefficient which is called the diffusion coefficient.

The equation which describes the diffusion in 1D is given by:

\[
\frac{\partial c}{\partial t} = -\frac{\partial J}{\partial x} = D \frac{\partial^2 c}{\partial x^2}
\]  

(24.21)

in 2D

\[
\frac{\partial c}{\partial t} = -\frac{\partial J}{\partial x} - \frac{\partial J}{\partial y} = D \frac{\partial^2 c}{\partial x^2} + D \frac{\partial^2 c}{\partial y^2}
\]  

(24.22)

24.3 Numerical study of PDEs

24.3.1 approximations of derivatives

In order to study PDE numerically we need to find numerical approximations for the main spatial and time derivatives of the function \( c \) and other functions which are used in our equations. Let us formulate a procedure for numerical solution of an PDE similar to the diffusion equation in 1D space (24.21). We need to specify in which region \( R \) of space in which we want to solve our equation, and we need to specify the initial distribution of the substance \( c = c_0(x) \).

After that we can discretize our equation in the region \( R \). This means the following. Assume that our region \( R \) is an interval \( 0 \leq x \leq L \). Let us place \( N \) equally spaced points at the distance
From each other to this region \( R \) (fig. 24.3). Now, instead of using a continuous function \( c(x) \) we use discrete values of \( c \) at that \( N \) points (fig. 24.3). Let us find a derivative of our discrete function \( c \) with respect to \( x \) at some point \( x_i \) which corresponds to an index \( i \) in fig. 24.3. If we denote the value of \( c(x_i) = c_i \), the value of \( c(x_{i+1}) = c(x_i + hx) = c_{i+1} \) and \( c(x_{i-1}) = c(x_i - hx) = c_{i-1} \). Using Taylor expansion we can write:

\[
\begin{align*}
    c(x_i + hx) &= c_{i+1} = c_i + c'(x_i)(x_i + hx - x_i) = c_i + c'(x_i)hx \\
    c(x_i - hx) &= c_{i-1} = c_i + c'(x_i)(x_i - hx - x_i) = c_i - c'(x_i)hx
\end{align*}
\]

from here we find that following approximations for the first derivative of our function with respect to \( x \) at the point \( x_i \):

\[
c'(x_i) = \frac{c_{i+1} - c_i}{hx}
\]

and

\[
c'(x_i) = \frac{c_{i+1} - c_{i-1}}{2hx}
\]

Both formulae (24.24) and (24.25) can be used for numerical procedures. It can be shown that approximation (24.25) is slightly more accurate.

In order to find the second derivative we use the Taylor approximations of the neighboring points up to the second order:

\[
\begin{align*}
    c_{i+1} &= c_i + c'(x_i)hx + \frac{1}{2}c''(x_i)hx^2 \\
    c_{i-1} &= c_i - c'(x_i)hx + \frac{1}{2}c''(x_i)hx^2 \\
    c_{i+1} + c_{i-1} &= 2c_i + c''(x_i)hx^2
\end{align*}
\]

from here we find that the second derivative of our function with respect to \( x \) at the point \( x_i \) is given by:

\[
c''(x_i) = \frac{c_{i+1} + c_{i-1} - 2c_i}{hx^2}
\]

**24.3.2 integration of an ODE**

Let us consider first a more simple problem: the procedure for integration of an ODE:

\[
\frac{dc}{dt} = f(c); \quad c(0) = c_0
\]

Because our solution is \( c(t) \) we can also discretize this function in \( N \) points and find the derivative using formulae (24.24). However because here we have a time derivative let us denote the step in our approximation as \( ht \) instead of \( hx \). We get the following approximation:

\[
\frac{dc}{dt}(t_i) = \frac{c_{i+1} - c_i}{ht} = f(c_i)
\]
or
\[
cia_{i+1} = c_i + htf(c_i) \tag{24.30}
\]

This formula gives us a key to the numerical solution of our equation. Because we know \( c \) at time 0 we can find \( c_1 \) from (24.30). Now, if we know \( c_1 \) we can find \( c_2 \) from (24.30), etc. So, in such a way we can find the complete solution of our ODE.

For numerical study we need to have initial conditions for our equation, we need to choose an integration step and define the output. We can avoid using many points \( i \) for \( c \). This is because at each step we need just two values of the variable: the value at the current moment of time \( c_i \) and values at the next moment of time \( c_{i+1} \). We denote the current value of the variable \( c_i \) as \( c \) and the new value of the variable \( c_{i+1} \) as \( d \). The program can be written as follows:

\[
\begin{align*}
\text{init} - \text{cond}() \\
\{ \\
\text{c} = c_0; \\
\} \\
\text{step}() \\
\{ \\
\text{d} = c + htf(c); \\
\text{c} = d \\
\} \\
\text{main}() \\
\text{init} - \text{cond}(); \\
\text{t} = 0. \\
\text{for} (\text{t} = 0; \text{t} < T; \text{t} = \text{t} + h) \\
\{ \text{step}(); \} \\
\text{print}(\text{c}); \\
\}
\end{align*}
\tag{24.31}
\]

\subsection*{24.3.3 integration of a PDE}

Let us now solve an PDE:
\[
\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} + f(c) \tag{24.32}
\]
We solve it in the region
\[
0 \leq x \leq L \tag{24.33}
\]
and the initial distribution of the substance \( c \) is:
\[
c(0,x) = c_0(x) \tag{24.34}
\]

Now because \( c \) depends on \( x \) we cannot use just one variable \( c \) for our equation, we need to introduce an array \( c[i] \) which contains \( N \) elements. Using this array we can approximate the second spatial derivative in (24.32). For the time derivative we use the same approach as in the previous section. We denote the new value of our variable as \( d[i] \). Therefore the time derivative can be written as:
\[
\frac{\partial c}{\partial t} = \frac{d[i] - c[i]}{ht} \tag{24.35}
\]
and the space derivative can be found using (24.27) as follows:
\[
D \frac{\partial^2 c}{\partial x^2} = D \frac{c[i+1] + c[i-1] - 2c[i]}{hx^2} \tag{24.36}
\]
Now we can find a new value of our variable at each point $d[i]$ if we know the old values $c[i]$ as follows:

$$d[i] = c[i] + ht \left( D \frac{c[i + 1] + c[i - 1] - 2c[i]}{hx^2} + f(c[i]) \right)$$

(24.37)

The program can be written as follows:

```c
init - cond()
{
    for(i = 0; i < (N - 1)i = i + 1){
        c[i] = c0[i];
    }
}
step()
{
    /* main loop */
    for(i = 0; i < (N - 1)i = i + 1){
        d[i] = c[i] + ht * (D * (c[i - 1] + c[i + 1] - 2*c[i]) / (H X H X) + f(c[i]));
    }
    /* shift of variable */
    for(i = 0; i < (N - 1)i = i + 1){
        c[i] = d[i];
    }
}
main()
init - cond();
t = 0.
for(t = 0; t < T; t = t + ht)
{
    step();
    for(i = 0; i < (N - 1)i = i + 1){
        print(c[i]);
    }
}
```

(24.38)

Note, that the procedure which is shown here is not complete. We can note that we have a trouble in subroutine `step if i = 0`, because in that case we need to substitute something instead of $c[i - 1] = c[-1]$ but we have no elements with such a number. In order to resolve this problem we need to add more information to our system. This information is called the boundary conditions. They show what happens at the boundary of our medium. For example for equation (24.32) we have two boundaries: $x = 0$ and $x = l$. It is reasonable to assume that we have no flux at that boundaries. Which formally means that

$$\frac{\partial c}{\partial x}(0) = 0; \quad \frac{\partial c}{\partial x}(L) = 0;$$

(24.39)

Numerically this means the following. If we use formula (24.24) for the second boundary condition we can write:

$$\frac{\partial c}{\partial x}(L) = (c[N + 1] - c[N]) / H X = 0; \quad c[N + 1] = c[N]$$

(24.40)

So it means that we can formally use a non-existing element $c[N + 1]$ and we can put there the value $c[N]$. Similarly for the first boundary we get:

$$c[-1] = c[0]$$

(24.41)
Therefore we can modify our `step()` subroutine in the following way:

```c
step()
{
    /* boundary conditions */
    c[-1] = c[0];
    c[N + 1] = c[N];
    /* main loop */
    for (i = 0; i < (N - 1); i = i + 1){
        d[i] = c[i] + ht * (D*(c[i-1] + c[i+1] - 2*c[i])/(H*H) + f(c[i]));
    }
    /* shift of variable */
    for (i = 0; i < (N - 1); i = i + 1){
        c[i] = d[i];
    }
}
```

(24.42)

Using similar programs we can study PDEs numerically.
Chapter 25

Turing instability

Consider a planar system of PDE:

\[
\begin{align*}
\frac{\partial A}{\partial t} &= D_a \Delta A + f_1(A, I) \\
\frac{\partial I}{\partial t} &= D_i \Delta I + f_2(A, I)
\end{align*}
\] (25.1)

Fixed point of the system without diffusion is:

\[
\begin{align*}
f_1(A^*, I^*) &= 0 \\
f_2(A^*, I^*) &= 0
\end{align*}
\] (25.2)

Stability can be found from the Jacobian:

\[
J = \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\] (25.3)

The conditions \( \text{Tr}J < 0 \) and \( \text{Det}J > 0 \) give the following stability conditions:

\[
\begin{align*}
a_{11} + a_{22} &< 0 \\
a_{11} a_{22} - a_{21} a_{12} &> 0
\end{align*}
\] (25.4)

To study stability of this equilibrium in the system with diffusion we introduce new variables:

\[
\begin{align*}
A &= A^* + a \\
I &= I^* + i
\end{align*}
\] (25.5)

Then the system becomes:

\[
\begin{align*}
\frac{\partial a}{\partial t} &= D_a \Delta a + f_1(A^* + a, I^* + i) \\
\frac{\partial i}{\partial t} &= D_i \Delta i + f_2(A^* + a, I^* + i)
\end{align*}
\] (25.6)

And the linearization of this system is:

\[
\begin{align*}
\frac{\partial a}{\partial t} &= D_a \Delta a + a_{11} a + a_{12} i \\
\frac{\partial i}{\partial t} &= D_i \Delta i + a_{21} a + a_{22} i
\end{align*}
\] (25.7)

We use the following perturbations:

\[
\begin{align*}
a(x, t) &= A(t) \cos kx \\
i(x, t) &= I(t) \cos kx
\end{align*}
\] (25.8)
After this substitution we get the following system of ODEs:

\[
\begin{align*}
\frac{\partial A}{\partial t} &= -k^2 D_a A + a_{11} A + a_{12} I \\
\frac{\partial I}{\partial t} &= -k^2 D_i I + a_{21} A + a_{22} I
\end{align*}
\]

(25.9)

Its Jacobian is:

\[
J = \begin{pmatrix}
-D_a k^2 + a_{11} & a_{12} \\
a_{21} & -D_i k^2 + a_{22}
\end{pmatrix}
\]

(25.10)

We see that the trace of this matrix is always negative

\[\text{Tr}J = -D_a k^2 + a_{11} - D_i k^2 + a_{22} < 0\]

(25.11)

because of the condition (4). Therefore the only possibility for instability is \(\text{Det}J < 0\), or:

\[(-D_a k^2 + a_{11}) \ast (-D_i k^2 + a_{22}) - a_{12} * a_{21} < 0\]

(25.12)

Let us replace \(k^2 = x\), then the last condition becomes:

\[(-D_a x + a_{11}) \ast (-D_i x + a_{22}) - a_{12} * a_{21} < 0\]

(25.13)

The left hand side here is just a quadratic polynomial. Let us find its minimum. The derivative of the left hand side function is:

\[2D_a D_i x_{min} - D_a a_{22} - D_i a_{11} = 0\]

(25.14)

\[x_{min} = \frac{D_a a_{22} + D_i a_{11}}{2D_a D_i}\]

(25.15)

and the minimal value is:

\[f_{min} = \frac{(D_a a_{22} + D_i a_{11})^2}{4D_a D_i} - \frac{(D_a a_{22} + D_i a_{11})^2}{2D_a D_i} + a_{11} * a_{22} - a_{21} * a_{12} < 0\]

(25.16)

so the condition for instability is:

\[\frac{(D_a a_{22} + D_i a_{11})^2}{4D_a D_i} > a_{11} * a_{22} - a_{21} * a_{12}\]

(25.17)

or

\[D_a a_{22} + D_i a_{11} > 2 \sqrt{D_a D_i} \ast (a_{11} * a_{22} - a_{21} * a_{12})\]

(25.18)

Or the overall conditions for Turing instability are:

\[
\begin{align*}
a_{11} + a_{22} &< 0 \\
a_{11} * a_{22} - a_{21} * a_{12} &> 0 \\
D_a a_{22} + D_i a_{11} &> 2 \sqrt{D_a D_i} \ast (a_{11} * a_{22} - a_{21} * a_{12})
\end{align*}
\]

(25.19)

In order to make conclusions on the conditions for the Turing instability, we rewrite the last condition in a more soft form as:

\[D_a a_{22} + D_i a_{11} > 0\]

(25.20)
Because $a_{11}$ account for autocatalisis $a_{11} >$, on the other hand from the first condition from (25.19) the sum $a_{11} + a_{22}$ must be negative, therefore $a_{22} < 0$. So we can rewrite (25.20) as

$$-D_a |a_{22}| + D_i |a_{11}| > 0$$

(25.21)

This yields

$$\frac{D_i}{|a_{22}|} > \frac{D_a}{|a_{11}|}$$

SO we see, that the Turing instability occurs if the diffusion coefficient of the inhibitor is larger that the diffusion coefficient of the activator.