1D equations: \( dx/dt = f(x, c) \)

Equilibrium \( dx/dt = f(x^*, c^*) = 0 \)

Hyperbolic if \( \frac{\partial f}{\partial x}(x^*) \neq 0 \)

Fold bifurcation:

Necessary conditions (NC): \( \frac{\partial f}{\partial x}(x^*, c^*) = 0 \)

Non-degenerate conditions (NND):

\[ \frac{\partial f}{\partial x}(x^*, c^*) \neq 0 \quad \frac{\partial^2 f}{\partial x^2}(x^*, c^*) \neq 0 \]

Normal form:

\[ d\eta/dt = \pm \mu \pm \eta^2 \]

Bifurcation diagrams

\[ \eta' = \mu - \eta^2 \]

\[ \eta' = -\mu + \eta^2 \]

2D equations:

\[ \begin{align*}
    dx/dt &= f(x, y, c) \\
    dy/dt &= g(x, y, c)
\end{align*} \]

Equilibrium in \( f(x^*, y^*, c^*) = 0 \), \( g(x^*, y^*, c^*) = 0 \)

Jacobian: \( J = \left( \begin{array}{cc}
    \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
    \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
  \end{array} \right) \) evaluated at the equilibrium.

For real eigenvalues \( (\lambda_1, \lambda_2): \) \( \text{tr}J = \lambda_1 + \lambda_2, \text{det}J = \lambda_1 \lambda_2 \)

For complex eigenvalues \( (\lambda_{1,2} = \alpha \pm i\beta, \alpha \neq 0): \) \( \text{tr}J = 2\alpha, \text{det}J = |\lambda|^2 = \alpha^2 + \beta^2 \)

Equilibrium type can be determined form the \textbf{det-tr}, as shown in the figure below:

- \textbf{det A}:
  - 3: stable spiral
  - 4: non-stable spiral
  - 5: center
  - 6: non-stable node

- \textbf{tr A}:
  - 1: saddle

Fold bifurcation:

NC: \( \lambda_1 = 0 \), or \( \text{det}J = 0 \).

NND: \( \lambda_2 \neq 0 \), or \( \text{tr}J \neq 0 \), addition NNDs exist, but not given.

Normal form on a center manifold the same as for 1D case.

Bifurcation diagrams, same as in 1D

Hopf bifurcation:

NC: \( \lambda = \alpha(c) \pm \beta(c); \) \( \alpha(c^*) = 0, \) or \( \text{tr}J = 0 \) and \( \text{det}J > 0 \)

NND: \( \alpha'(0) \neq 0; \) \( \beta(0) \neq 0; \) and \( I_{\text{stability}} \neq 0 \)

Normal form in polar coordinates: \( \dot{r} = \pm ry \pm r^3 \)

Bifurcation diagrams

\[ \begin{align*}
    \dot{r}/dt &= \gamma r + r^3 \\
    \dot{r}/dt &= -\gamma r + r^3
  \end{align*} \]

Stability index:

For the system \( \begin{cases} 
    dx/dt = A + \omega y + Y^1 \\
    dy/dt = B - \alpha x + Y^2 \\
    I = \omega(Y^1_{xx}x^2 + Y^1_{xy}y_x + Y^2_{xy}y_x + Y^2_{yy}y_y) \\
    + (Y^1_{xx}x^2 - Y^1_{xy}y_x + Y^2_{xy}y_x + Y^2_{yy}y_y) \\
    + (Y^1_{yy}y_y - Y^1_{xy}y_x + Y^2_{xy}y_x + Y^2_{yy}y_y) \\
    \end{cases} \)

where \( Y^1_{xy}(0, 0) = \frac{\partial^2 Y^1}{\partial y \partial x}(0, 0); \) \( Y^2_{xy}(0, 0) = \frac{\partial^2 Y^2}{\partial y \partial x}(0, 0); \) etc

For the system \( \begin{cases} 
    dx/dt = A - \omega y + Y^1 \\
    dy/dt = B + \alpha x + Y^2 \\
    I = \omega(Y^1_{xx}x^2 + Y^1_{xy}y_x + Y^2_{xy}y_x + Y^2_{yy}y_y) \\
    + (Y^1_{xx}x^2 - Y^1_{xy}y_x + Y^2_{xy}y_x + Y^2_{yy}y_y) \\
    + (Y^1_{yy}y_y - Y^1_{xy}y_x + Y^2_{xy}y_x + Y^2_{yy}y_y) \\
    \end{cases} \)

If the index \( I \) is negative, then equilibrium is stable.

Note, that for the linear system \( \text{det}A = ad - bc, \text{tr}A = a + d, \) \( D = (\text{tr}A)^2 - 4 \cdot \text{det}A \)

And \( D < 0 \) above the parabola on the figure.

Equilibrium is hyperbolic if all the eigenvalue of the Jacobian matrix have nonzero real parts: i.e. for real \( \lambda_1 \neq 0, \lambda_2 \neq 0, \)

for complex \( \lambda = \alpha \pm i\beta, \alpha \neq 0. \)