1 Exponentials

Imagine a single bacterium that divides into two. These two divide into four, and those into eight - and so on, until after n divisions, if all survive, there are 2^n bacteria. This is quiet easy to calculate, and indeed a good example of simple use of exponentials in biology.

Next, we may ask how many generations would be needed to obtain say 10^9 bacteria starting from a single one and given that none dies. In other words, what must n be for 2^n to approximate 10^9. Here are few rules for working with exponentials, that you can use to be able to estimate this number without a calculator:

- $a^n + a^m \neq a^{n+m}$ but $a^n \times a^m = a^{n+m}$
- $a^n : a^m = a^{n-m}$ if $a \neq 0$
- $(a^n)^m = a^{nm}$
- $a^n b^n = (ab)^n$
- $a^m / b^m = (a/b)^m$
- $a^{-n} = 1/a^n$
- $a^0 = 1$

We know that $2^4 = 16$. Using the rules above we get $2^8 = 16 \times 16 = 256$ and $2^{10} = 256 \times 4 = 1024$. 1024 is a number close to 1000. Thus to reach 10^9 we need approximately $1024^3 = (2^{10})^3 = 2^{30}$. Therefore probably 30 generations would be enough to produce 10^9 bacteria. Checking with a calculator, you will see that indeed $2^{30} = 1,073,741,824$.

2 Logarithms

Suppose now that we want to plot the growth of the bacteria with respect to the generation number, i.e., we want to plot 2^n versus n. Since 2^n grows much faster than n, plotted in the usual way either the highest values lie off the paper, or the lowest values are hard to distinguish from zero. The same difficulty arises with, for example, graphs of masses of mammals, as these range in size from a few grams (shrews) to 10^5 kg (a whale).

The usual solution to this common problem is to use a scale where we express numbers in powers of 10, i.e., $10^0, 10^1, 10^2, 10^3$ etc and plot the power of 10, ie. $0, 1, 2, 3$, rather than the number itself, $10^0, 10^1, 10^2, 10^3$ on the axis. The scale can be extended using negative exponentials as well, i.e., $10^{-1}, 10^{-2}$ to express the numbers smaller than 1. This is a scale that easily expands many "orders of magnitude". This expression is widely used to compare quantities, especially when they are not exactly known. To say that A is one order of magnitude bigger than B indicates that A is almost 10 ten times B.

On this scale the numerals are "logarithms with base 10" of the numbers. Thus, the logarithm of $10,000$ or 10^4, is 4. In general, we express this as follows: $log_{10}x = k$ if and only if $10^k = x$. Because $1 = 10^0$,

$log1 = 0$. Similarly, $log10 = 1$.

Logarithms, just like exponents, can have different bases. In the biological sciences, you are likely to encounter the base 10 logarithm, known as the common logarithm and denoted simply as log; and the base e logarithm, known as the natural log and denoted as ln. That is: $log_e x = lnx = k$ if and only if $e^k = x$. Most calculators will easily compute these widely used logarithms. Logarithms are quite convenient for biologists who work over many orders of magnitude and on many different scales, because they transform exponentially increasing inputs into linearly increasing outputs. Remember: exponential and logarithmic functions are inverses of each others. This is true of all logarithms, regardless of base.

The base b logarithm of a positive number x is the exponent you get when you write x as a power of b where $b > 0$ and $b \neq 1$. That is, $log_b x = k$ if and only if $b^k = x$.

Examples:

$2^4 = 16 \longleftrightarrow log_2(16) = log_2(2^4) = 4$

$10^3 = 1000 \longleftrightarrow log_{10}(1000) = log_{10}(10^3) = 3$

$e^4 \approx 54.598 \longleftrightarrow log_e(e^4) = ln(e^4) = 4$

To convert from one base to the other, use the formula:
$log_a(b) = log(b)/log(a)$.

Finally, let us go back to our initial problem. We wanted to plot the growth of the bacteria. If we plot n versus 2^n using a linear ("normal") scale for n and a logarithmic one for 2^n, the graph should look like a line. This is because in this case the bacteria are growing exponentially. Often biologists are interested in knowing whether or not a population is growing exponentially. Plotting the data you have available on a log-linear scale (e.g., the year in the linear scale and the size of the population in the logarithmic scale) will give you a very fast insight into this question:

- If the population is growing exponentially, you will see a straight line.
- If the growth is slower than the exponential, the curve will be convex.
- if the growth is faster than exponential, the curve will be concave.

2.1 Properties of logarithms

If logs have the same base:

- $log(a) + log(b) = log(ab)$
- $log(a) - log(b) = log(a/b)$
- $a * log(b) = log(b^a)$

$log_a(c) = b$ is inverse of $a^b = c$

Logs and exps cancel each other:
$log_{10}(10^a) = a$
$e^{ln(b)} = b$

Example:

Solve with rules for logs and/or with rules for exps the expression $e^{(a*ln(b/c))}$.
$e^{a*ln(b/c)} = e^{ln(b/c)^a} = (b/c)^a$ or $e^{a*ln(b/c)} = (e^a)^{ln(b/c)} = (e^{ln(b/c)})^a = (b/c)^a$.

3 Online studiemateriaal

3.1 Khan Academy

On the website of the Khan Academy (http://www.khanacademy.org/) there are a number weblectures about calculating exponentials. Because weblectures still are re-located, it is most convenient to go to the main website and use the search function. You can simply use the search term ”exponentials” or ”logarithms”. You will see that next to the lectures, there are also exercises available in Khan academy.

3.2 DWO

More exercises are available in the DWO package of the Freudenthal Institute. You can login as guest at http://www.fi.uu.nl/dwo/sk. Or you can register: this will allow you to save your work, ie next time you can see which exercises you could make and which you could not. Choose for module E “exponentials and logarithms”.

3.3 Exercises

a. $7^{345} \times 7^5 =$
b. $32^{20} \times 32^{18} =$
c. $2^5 \times 2^2 \times 2^3 =$
d. $3^{25} \times 3^{25} \times 3^{25} =$
e. *Bacillus cereus* divides every 30 minutes. You inoculate a culture with exactly 100 bacterial cells. After 3 hours, how many bacteria are present?
f. A colony of bacteria is growing under ideal conditions in a laboratory. At the end of 3 hours there are 10,000 bacteria and at the end of 5 hours there are 40,000. How many bacteria were present initially?
g. Simplify the following expression, assume that $n \neq 0$ and $p \neq 0$:
$\frac{(n^{-3})^4}{(n^4 p^{-3})^{-3}}$.
h. $log_8 512 =?$
i. if $log(2x) = 3, x =?$

j. if $5e^x = 11, x = ?$

k. if $log_2(64^x) = 36, x = ?$

l. Prove or disprove that if $e^{-ax} = \frac{1}{2}^{bx}$, then $b = \frac{a}{ln2}$.