1 Algebraic equations

Algebraic expressions

Algebraic expressions may contain numbers, variables, parameters and arithmetic operations. Below, we review examples of several basic operations which help us to work with algebraic expressions.

One of the most basic algebraic operations is getting rid of parentheses to simplify the expression. For that we use the following rule:

\[(a + b)(c + d) = ac + ad + bc + bd\]

note, that here \(ac\) means \(a \times c\), etc., as the multiplication sign is often omitted.

Example (open parenthesis): \((4x + 2a)(2 - 3x) = 8x - 12x^2 + 4a - 6ax\)

On the other hand, we also often introduce parentheses to decompose an expression into multiple factors that then can be solved separately:

Example (factor expression): \(9x^3 + 3x^2 - 6a^4x^2 = x^2(9x + 3 - 6a^4) = 3x^2(3x + 1 - 2a^4)\)

Solving of equations

An equation is a mathematical relationship involving one or more unknown variables. Solving equations means finding the values of these unknowns such that after substitution of these values back into the equation the left and right hand sides will be equal to each other. For example: the equation \(2x - 16 = -10\) has as a solution \(x = 3\), as \(2 \times 3 - 16 = 6 - 16 = -10\).

Solving equations containing a single value means finding one or more values for the variable for which the equation holds true. In order to find these solutions we need to move all terms containing the variable to one side of the = sign. This can be achieved by multiplying the left and right hand side of the equation by the same amount, or by adding or subtracting the same amount from the left and right hand side of the equation. Remember that for quadratic equations of the form \(ax^2 + bx + c = 0\) we can use the so-called 'abc' formula, which gives us the two possible solutions as \(x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\).

Example: solve a simple equation with one variable: \(2x - 3 = 0\) So \(2x = 3\), so \(x = 1.5\).

Example: solve a more complex equation with one variable: \(aN^2 = bN\)

First bring all terms with \(N\) to one side, so \(aN^2 - bN = 0\), than factor out \(N\), so \(N(aN - b) = 0\). Now realize that if \(A \times B = 0\) either \(A = 0\) or \(B = 0\). So, for our equation \(N = 0\) or \(aN - b = 0\). So finally, \(N = 0\) or \(N = \frac{b}{a}\).

If an equation contains multiple variables, you can choose which variable you will put on its own to one side of the = sign. You then solve the equation and express the solution in terms of this particular variable. Which variable you choose for this is up to you, so it is best to pick the variable for which solving the equation is easiest. The solution of the equation then describes the value of this variable as a function of the values of the other variables that were in the equation. Try this with the following examples:

Example: solve the equation for one of the two variables: \(aN - bM^2 = 0\)

Note that \(N\) is more simple to solve for than \(M\), so \(aN = bM^2\) and hence \(N = \frac{b}{a}M^2\).
Example: solve the equation for one of the two variables: \( aN = b^M \)
Note that now \( M \) is more simple to solve for than \( N \). First multiply left and right with \( N \). This give us \( aN^2 = bM \) and hence \( M = \frac{a}{b}N^2 \).

Many complex equations can not be solved analytically. As an example, higher order equations, also called polynomials, of the form \( a + bx + cx^2 + dx^3 + \cdots = 0 \) can only be solved if we can factor them into multiple terms of lower order that then each can be solved separately. Sometimes another option is to introduce a new variable \( y = x^\nu \), if all terms of the equation can be somehow rewritten as \( a^y \). This gives you a simpler equation in \( y \) that now can be solved, this solution can then subsequently be filled in to solve \( x \) itself.

Example: solve the following equation: \( aN^6 - bN^3 = 0 \).
Note that both terms can be written in terms of \( N^3 \), so we can introduce \( M = N^3 \) and rewrite the equation as \( aM^2 - bM = 0 \). We already know how to solve this: \( M(aM - b) = 0 \) and hence \( M = 0 \) or \( M = \frac{b}{a} \). Therefore \( N^3 = 0 \) and thus \( N = 0 \), or \( N^3 = \frac{b}{a} \) and thus \( N = \left(\frac{b}{a}\right)^{1/3} \)

Solving systems of equations

Solving systems of coupled algebraic equations works in principle the same as solving several independent equations. However, an important difference is that solutions of one equation are substituted into other equations to help find their solution.

As an example, consider the equations \( 3x + 2y = 0 \) and \( y/5 = 3 \). Solving the first equation gives you \( 3x = -2y \) and hence \( x = -(2/3)y \). Solving the second equation gives you \( y = 15 \). If these two equations form a system, that is, they are coupled and hence the \( y \) in the first and second equation are the same, we can fill in the \( y = 15 \) we obtained from solving the second equation into \( x = -(2/3)y \) we obtained from solving the first equation and obtain \( x = -10 \) and hence find the final solution of the first equation. Note that if you are solving a system of two or more algebraic equations it is not necessarily the best approach to start with the first equation and work your way down. Instead, the smart approach is to start with the equations that are most easy to solve. Indeed, in the example the second equation is much easier than the first, and we used it’s solution to find the final solution of the first equation. Furthermore, only fill in a solution of one equation into a solution of another equation if this actually simplifies matters.

Example 1: solve the following system of equations: \( 0 = aN - bM \) and \( 0 = M(cN - d) \)
From the first equation we obtain \( N = \frac{b}{a}M \). From the second equation we obtain \( M = 0 \) or \( cN - d = 0 \) and hence \( N = \frac{d}{c} \). If we now combine this with the solution of the first equation, \( M = 0 \) gives us \( N = 0 \) and \( N = \frac{d}{c} \) gives us \( \frac{d}{c} = \frac{b}{a}M \) and hence \( M = \frac{dc}{ab} \). So we have two solutions \( N = 0, M = 0 \) and \( N = \frac{d}{c}, M = \frac{dc}{ab} \).

Example 2: solve the following system of equations for \( n \) and \( p \): \[
\begin{align*}
an - an^2 - bn p &= 0 \\
np - kp &= 0
\end{align*}
\]
Let us start with the second equation, as it looks easier. \( np - kp = 0 \) gives us \( p(n-k) = 0 \) and hence \( p = 0 \) or \( n = k \). Now let us move back to the first equation, \( an - an^2 - bnp = 0 \). First, we fill in \( p = 0 \). This gives us \( an - an^2 = 0 \) and hence \( an(1 - n) = 0 \) and hence \( n = 0 \) or \( n = 1 \). This means that from filling in only the first solution of the first equation we already obtained two overall solutions for this system \( n = 0, p = 0 \), and \( n = 1, p = 0 \). Let us also fill in the second solution of the second equation \( (n = k) \) into the first equation, this gives us \( ak - ak^2 - bkp = 0 \). Note that as \( k \) is a parameter, \( k = 0 \) is not a solution!. However, we can divide all by \( k \), resulting in \( a - ak - b \frac{p}{k} = 0 \) or \( a - ak = bp \) and hence \( p = \frac{a-ak}{b} \). The third overall solution thus is \( n = k, p = \frac{a-ak}{b} \). Summarizing all solutions, we found the
following \((n, p)\) values as possible solutions to the given system of equations: \((0, 0), (1, 0), (k, \frac{a-ak}{p})\).

## 2 Online studiemateriaal

### 2.1 Khan Academy

On the website of the Khan Academy (http://www.khanacademy.org/) there are a number of lectures about calculating exponentials. Because weblectures are re-located, it is most convenient to go to the main website and use the search function. Section on ”Creating and solving linear equations” is a large one but it covers all the material you need for working with equations. You will see that next to the lectures, there are also exercises available in Khan academy.

### 2.2 DWO

More exercises are available in the DWO package of the Freudenthal Institute. You can login as guest at http://www.fi.uu.nl/dwo/sk. Or you can register: this will allow you to save your work, ie next time you can see which exercises you could make and which you could not. Choose for module A for “Algebraische vaardigheden” and then continue to ”Oefenen met vergelijkingen 1-4”.

## 3 Exercises

1. Simplify the below expressions:
   
   (a) \((ax - 2by)(3y - 4bx) + 2b(2ax^2 + 3y^2) - 8xyb^2\)
   
   (b) \(\frac{6}{5} - \frac{5r}{30r+5}\)

2. Solve the equation for the specified variable:
   
   (a) find \(r\) in: \(3r + 2 - 5(r + 1) = 6r + 4\)
   
   (b) find \(x\) in: \(x + \frac{4}{x} = 4\)
   
   (c) find \(N\) in: \((b - \frac{N}{k})N = 0\)
   
   (d) find \(N\) in: \((b - d(1 + \frac{N}{k}))N = 0, d \neq 0; k \neq 0\)
   
   (e) find \(N\) in: \((\frac{b}{1+N/k} - d)N = 0, b \neq 0;\)

3. Solve the system of equations for the specified variables:
   
   (a) find \(x, y\) in: \(\begin{cases} x - 2y = -5 \\ 2x + y = 10 \end{cases}\)
   
   (b) find \(x, y\) in: \(\begin{cases} ax + by = 0 \\ cx + dy = -b \end{cases}\)
   
   (c) find \(x, y\) in: \(\begin{cases} x(1 - 2x) + xy = 0 \\ 4y - xy = 0 \end{cases}\)
   
   (d) find \(x, y\) in: \(\begin{cases} 4x - xy - x^2 = 0 \\ 9y - 3xy - y^2 = 0 \end{cases}\)
(e) find $R, N$ in:

\[
\begin{aligned}
\begin{cases}
  b(1 - \frac{g}{k} - d - aN)R &= 0 \\
  (R - \delta)N &= 0
\end{cases}
\end{aligned}
\]

, $a, b, d, k, \delta \neq 0$;