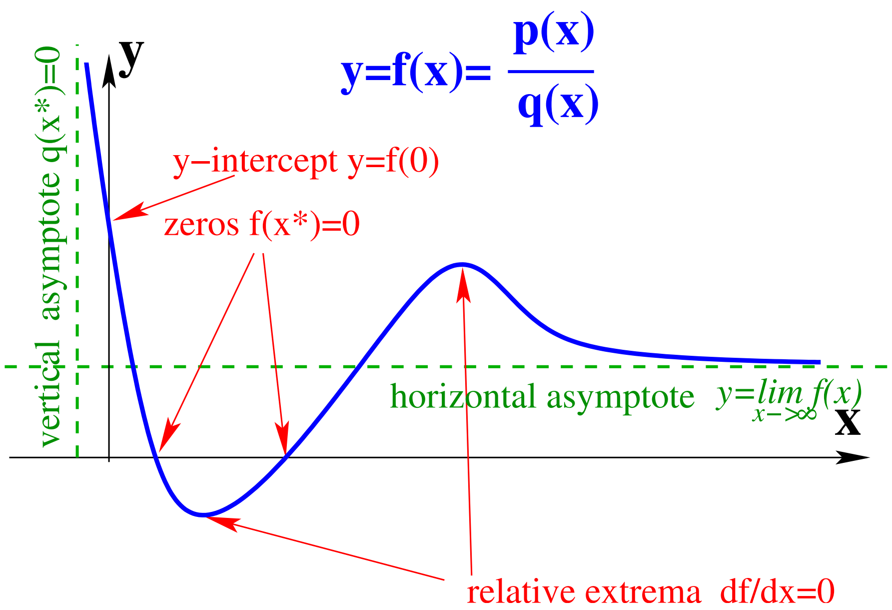
Basic Mathematics for Biologists



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Chapter 1

Introduction

In Biology, as is the case with many sciences, you will occasionally encounter mathematics. Wherever you go after your Biology degree, you are likely to encounter models – interesting simplifications that allow us to understand complex biological systems better. These models will often containing basic mathematical equations. The purpose of this reader is twofold:

1. Refresh your memory of high school mathematics
2. Getting used to parameters rather than specific values

A difference between the equations and functions used in high school and the ones in this course is the use of so-called **parameters** rather than numbers. As an example, consider the simple equation:

$$(3-x)(x+2)=0 \tag{1.1}$$

The left-hand side of the equation contains the **variable** x, and the numbers 3 and 2. By looking at the equation, you may notice that when x is 3, the first term (3-x) becomes 0. The second term (x+2) becomes 5. So when $x=3$, the equation becomes 0×5 = 0, which is true. So one **solution** to equation 1.1 is x=3. There is also a second solution, namely when x=-2.

As mentioned above, in this course we will often work with parameters. Instead of equation 1.1, you will encounter equations that look like this:

$$(a-x)(x+b)=0 \tag{1.2}$$

Unless explicitly stated differently, you may always assume that these parameters have a positive value. Although equation 1.2 looks very different than equation 1.1, do not let this scare you. The solutions are still very simple. The left-hand side of the equation is equal to 0 when x=a, or x=-b.

Another important difference with high school mathematics is that **variables** will not necessarily be called x, y or z, but may be assigned any letter to conveniently express what we study. For example, you may choose to describe the number of rabbits in a population as R, or the concentration of chemicals in a reaction c. As a result, you will have to learn to distinguish which characters represent the *variables* and which represent the free *parameters*. Remember that variables which have a **dynamical behaviour**, meaning that we want to understand how they change over time. In contrast, a free parameter reflects something that is fixed, for example the death rate or birth rate of individuals in a population. Its value is constant for a particular population, but could be different for different populations at different locations or for different species.

One benefit of working with parameters is that we can study the behaviour of our system for a range of values, rather than for a particular value. Moreover, we could potentially identify **parameter conditions** under which our system is viable. For example, it enables us to say “when $d > b$ (when death exceeds birth), the population will die out”.

A problem with using parameters rather than numbers is that you can not use your graphical calculator to draw functions or solve equations. Of course, you can always fill in some arbitrary numbers for the parameters and still use a graphical calculator (also see *www.desmos.com/calculator* for a good online graphical calculator). Note however that the general function or equation may have more than one solution, whereas by filling in a particular set of made up numbers you will always find only one of these solutions. As a very simple example, consider having to draw the line $y = ax + b$. This describes a simple straight line with slope a, intersecting the y-axis at point b. Thus, if we choose $a = 1$ and $b = 2$ we get a line with a positive slope and a positive intersection point with the y-axis. However, the general equation $y = ax + b$ also holds for lines with a negative slope ($a < 0$) and/or negative intercepts with the y-axis ($b < 0$). The more parameters you have, the less feasible it becomes to draw all versions of a mathematical function simply by filling in parameters in a graph. Instead, it is important to get adjusted to how to work with mathematical equations, and how to sketch their shapes. That is what this reader is for.

Chapter 2

Exponentials and Logarithms

2.1 Introduction to exponentials

Imagine a single bacterium that divides into two. These two divide into four, four divide into eight – and so on, until after n divisions, if all survive, there are 2^n bacteria. This is quite easy to calculate, and a good example of the relevance of exponentials in biology. To express this in a simple mathematical equation, we could define B as the number of bacteria, and a n to describe the number of divisions:

$$B = 2^n \tag{2.1}$$

Next, we may ask how many divisions would be needed to obtain, for example, 10^9 bacteria. In other words, we are asking n needs to be for 2^n to reach 10^9 or more bacteria. In the next section we provide you with the rules for working with exponentials, which you can use to be able to estimate this number without a calculator.

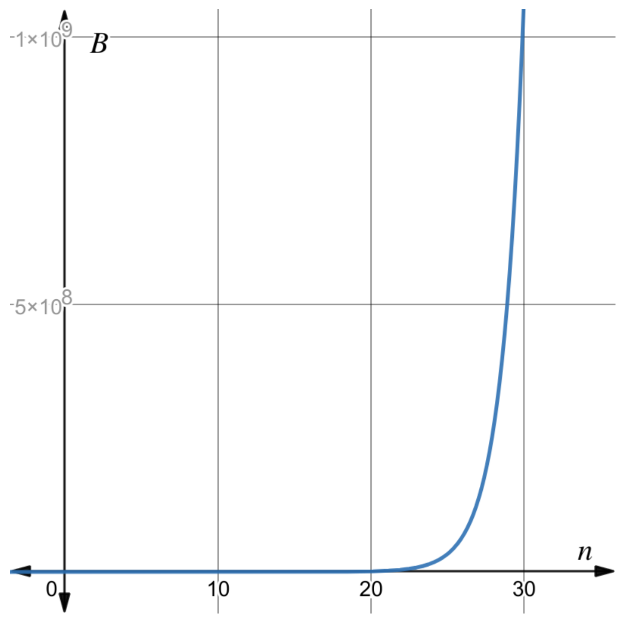
2.2 Rules to work with exponentials

- $a^n + a^m \neq a^{n+m}$ (a common beginners mistake)
- $a^n \times a^m = a^{n+m}$
- $a^n : a^m = a^{n-m}$ if $a \neq 0$
- $(a^n)^m = a^{nm}$
- $(a^n \cdot b^m)^q = a^{nq} \cdot b^{mq}$
- $a^n b^n = (ab)^n$
- $a^m / b^m = (a/b)^m$
- $a^{-n} = 1/a^n$
- $a^0 = 1$

Now, using the rules defined above, we can figure out after how many divisions we will have 10^9 bacteria. We know that $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$. We can also see that $2^8 = 2^4 \times 2^4 = 16 \times 16 = 256$. Applying the same trick again, and we will see that $2^{10} = 2^8 \times 2^2 = 256 \times 4 = 1024$. So without a calculator, we now know that 10 divisions lead to 1024 cells, a number close to 1000. Remember that one million (10^6) is a thousand times a thousand, and one billion (10^9) is a thousand times a million. So to get to a billion cells, we need to multiply 1000 by 1000, and then multiply the result by 1000 again. In other words, a billion cells is 1000^3. This we can approximate by raising the 1024 we found earlier to the power three: $1024^3 = (2^{10})^3$. Now, following one of the rules describe above, we know that $(2^{10})^3 = 2^{30}$. Therefore, 30 generations will produce 10^9 bacteria (or actually more, since $1024 > 1000$). Checking with a calculator, you will see that indeed $2^{30} = 1,073,741,824$. By simply following the basic rules of exponentials, we could get a very good approximation of how many divisions were necessary!

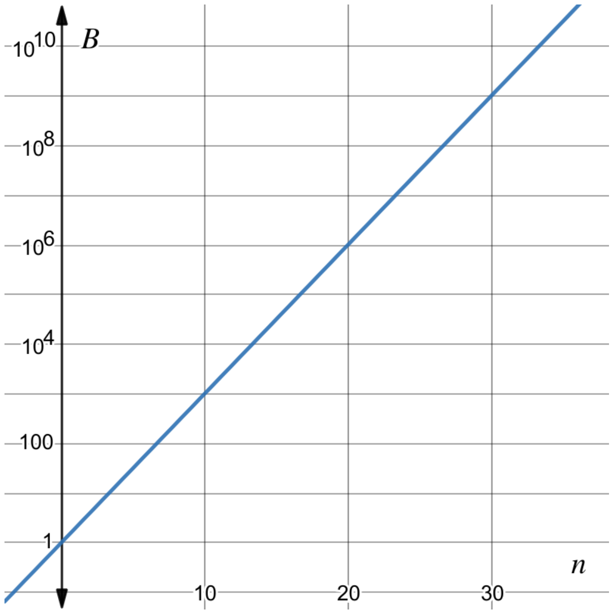
2.3 Introduction to logarithms

Suppose now that we want to plot the growth of the bacteria with respect to the generation number, i.e., we want to plot 2^n versus n. Since 2^n grows much faster than n, plotted in the usual way, the function would look like this:



Plotting $B = 2^n$ using linear axes, with n on the x-axis and B on the y-axis.

The problem with this graph is that the lowest values are hard to distinguish from zero, as the differences in n span **orders of magnitude**. Orders of magnitude are often used to refer to large differences in values. For example, saying that A is one order of magnitude bigger than B indicates that A is approximately 10 times larger than B, whereas two orders of magnitude would indicate a 100-fold difference. Clearly, when values differ orders of magnitude, difficulty arises in plotting them clearly. The same difficulty arises with, for example, with graphs of masses of mammals, as these range in size from a few grams (shrews) to 10^5 kg (a whale). The usual solution to this common problem is to use a scale where we express numbers in powers of 10, i.e., $10^0, 10^1, 10^2, 10^3$ etc and plot the power of 10, ie. $0, 1, 2, 3$, rather than the number itself, $10^0, 10^1, 10^2, 10^3$ on the axis. In the example above, if we draw the y-axis like this, the figure becomes:



Plotting $B = 2^n$ using a logarithmic y-axis, with n on the x-axis and $\log(B)$ on the y-axis.

From this graph, you can easily read that after 20 divisions (n=20, x-axis), we expect 10^6 (one million) cells, which was impossible to see from the previous graph. The scale can be extended using negative exponentials as well, i.e., $10^{-1}, 10^{-2}$ to express the numbers smaller than 1. Using this approach one can easily plot numbers ranging over many orders of magnitude in a single graph.

2.3.1 Logarithms with different bases

Above, we have discussed "logarithms with base 10", meaning that we express the original number as a number we need to raise 10 to. For example, if we want to express the number 10,000, we need to raise 10 to the power of 4. Thus, the logarithm of $10,000$ or 10^4, is 4. In general, we can state that $\log_{10} x = k$ if and only if $10^k = x$.

Logarithms, just like exponents, can have different bases. In the biological sciences, you are likely to encounter the base 10 logarithm as discussed above. This common logarithm is often simply written as log, rather than $\log_{10}$. Another common logarithm is $\log_e$, known as the natural log and often simply denoted as ln. That is: $\log_e x = \ln x = k$ if and only if $e^k = x$. Most basic calculators will easily compute these widely used logarithms.

Note that exponential and logarithmic functions are inverses of each other. This is true of all logarithms, regardless of base. Hence, the logarithm of base b of a positive number x is the exponent you get when you write x as a power of b where $b > 0$ and $b \neq 1$. That is, once again, $\log_b x = k$ if and only if $b^k = x$. Here, all numbers are replaced by parameters. I may look a bit scary, but rest assured: nothing has changed. If you have a hard time remembering these rules, simply memorise that $\log_{10} 100 = 2$ is equivalent to $10^2 = 100$. With this in mind, you can always reconstruct the conversion rules from exponentials to logarithms by replacing 10, 100, and 2 with the appropriate values.

Examples:

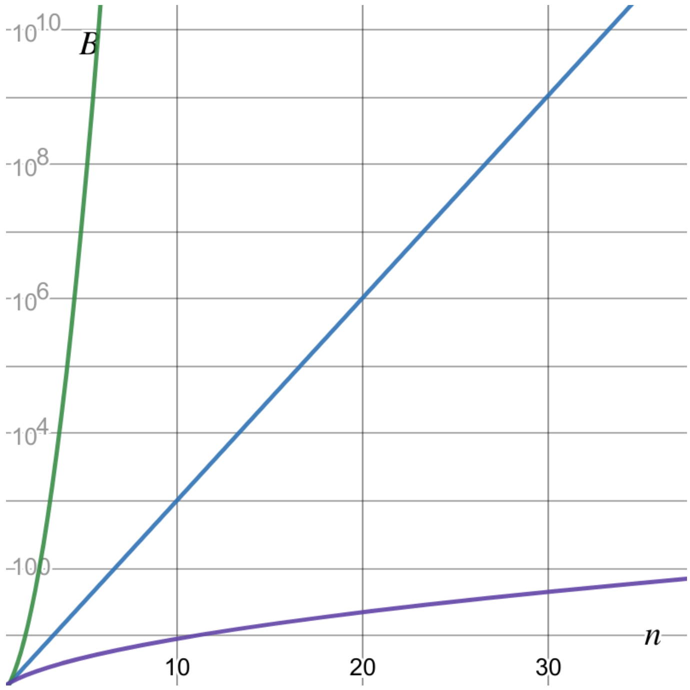
$2^4 = 16 \longleftrightarrow log_2(16) = log_2(2^4) = 4$

$10^3 = 1000 \longleftrightarrow log_{10}(1000) = \log_{10}(10^3) = 3$

$e^4 \approx 54.598 \longleftrightarrow log_e(e^4) = \ln(e^4) = 4$

Finally, let us go back to our initial problem. We wanted to plot the growth of the bacteria. If we plot n versus 2^n using a linear ("normal") scale for n and a logarithmic scale for 2^n, the graph should look like a straight increasing line. This is because the bacteria are growing exponentially. Often biologists are interested in knowing whether or not a population is growing exponentially. Plotting the data you have available on a log-linear scale (e.g., the year in the linear scale and the size of the population in the logarithmic scale) will allow you to answer this question:

- If the population is growing exponentially, you will see a straight line (blue line in figure below)
- If the growth is slower than the exponential, the curve will be concave (purple line)
- if the growth is faster than exponential, the curve will be convex (green line)



Plotting $B = 2^n$ using a logarithmic y-axis, with n on the x-axis and $\log(B)$ on the y-axis.

2.3.2 Rules to work with logarithms

If logs have the same base:

- $\log(a)+\log(b)=\log(ab)$
- $\log(a)-\log(b)=\log(a/b)$
- $a\cdot\log(b)=\log(b^a)$

Converting between exponentials and logarithms:

- $\log_a(c)=b$ is inverse of $a^b=c$

Converting to a different base:

- $\log_a(b)=\log_c(b)/\log_c(a)$

Logs and exps cancel each other:

- $\log_{10}(10^a)=a$
- $e^{\ln(b)}=b$

Example:

Solve with rules for logs and/or with rules for exps the expression $e^{(a\cdot\ln(b/c))}$.
$e^{a\cdot\ln(b/c)}=e^{\ln(b/c)^a}=(b/c)^a$ or $e^{a\cdot\ln(b/c)}=(e^a)^{\ln(b/c)}=(e^{\ln(b/c)})^a=(b/c)^a$.

2.4 Exercises

1. $7^{345}\times 7^5=$

2. $32^{20}\times 32^{18}=$

3. $2^5\times 2^2\times 2^3=$

4. $3^{25}\times 3^{25}\times 3^{25}=$

5. *Bacillus cereus* divides every 30 minutes. You inoculate a culture with exactly 100 bacterial cells. After 3 hours, how many bacteria are present?

6. A colony of bacteria is growing under ideal conditions in a laboratory. At the end of 3 hours there are 10,000 bacteria and at the end of 5 hours there are 40,000. How many bacteria were present initially?

7. Simplify the following expression, assume that $n \neq 0$ and $p \neq 0$:
$\frac{(n^{-3})^4}{(n^4 p^{-3})^{-3}}$.

8. $\log_8 512 =?$

9. if $\log(2x) = 3, x =?$

10. if $5e^x = 11, x =?$

11. if $log_2(64^x) = 36, x =?$

12. Prove or disprove that if $e^{-ax} = \frac{1}{2}^{bx}$, then $b = \frac{a}{\ln 2}$.

Chapter 3

Fractions

3.1 Introduction

Assume that you are an ecologist wishing to compare two different populations of rabbits. For one of the populations it is known that each year $\frac{1}{2}$ of the animals die because of disease and $\frac{1}{5}$ will be shot by hunters. For the second population it is only known that each year $\frac{4}{7}$ of the animals die. The birth rate is the same in the two populations. To determine which of the two rabbit populations is doing better you will need to compute whether $\frac{1}{2} + \frac{1}{5}$ is larger or smaller than $\frac{4}{7}$. Thus, to answer these types of questions it is important that you are capable of working with fractions. Below we discuss the general rules of working with fractions.

3.2 Rules to work with fractions

First some nomenclature. Consider a fraction $\frac{a}{b}$. Here a is called the **numerator** (*teller* in Dutch), and b is called the **denominator** (*noemer* in Dutch).

Simplifying a fraction:
The value of a fraction remains equal if numerator and denominator are multiplied by or divided by the same number.

$$\frac{ca}{cb} = \frac{a}{b}$$

$$\frac{a/c}{b/c} = \frac{a}{b}$$

Multiplication of two fractions:
Multiplying two fractions will generate as an answer a new fraction. Multiply the numerators of the two fractions to obtain the numerator of the new fraction. Multiply the denominators of the two fractions to obtain the denominator of the new fraction.

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

Division by a fraction
Division by a fraction is equal to multiplication by the inverse of that fraction.

$$a : \frac{b}{c} = a \times \frac{c}{b} = \frac{ac}{b}$$

$$\frac{a}{b} : \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

Addition of two fractions
Two fractions can only be added if they share the same denominator:

$$\frac{a}{b} + \frac{c}{b} = \frac{a+b}{c}$$

But what if the fractions do not have the same denominator? You can transform fractions into two fractions with the same denominator by multiplying both the numerator and denominator of the first fraction with the denominator of the second fraction, and by multiplying both the numerator and the denominator of the second fraction with the denominator of the first fraction:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{cb}{bd}$$

After this the fractions can simply be added by adding up the two numerators and dividing them by their shared denominator:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}$$

So, in summary: first do $\frac{a}{b} \times \frac{d}{d} = \frac{ad}{bd}$, then do $\frac{c}{d} \times \frac{b}{b} = \frac{cb}{bd}$, and then add them together $\frac{ad}{bd} + \frac{cb}{bd} = \frac{ad+cb}{bd}$

Note that applying the above rules in reverse means that $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$, so you can (when useful) split a fraction into two! However, you cannot do the same with an addition in the denominator: $\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}$

Note that all of the above also applies for subtraction of fractions the same rules apply as for addition of fractions: first the fractions need to share the same denominator, after this the numerators can simply be subtracted.

Removing a fraction from an equation
In equations we can often get rid of a fraction by multiplying both the left hand side (left of the = sign) and the right hand side (right of the = sign) with the denominator of the fraction.

$$\frac{a}{b} = cx$$

$$a = bcx$$

Example 1: solve the following equation: $\frac{2}{5x}+\frac{3}{4}=0.5$ First we ensure that the two fractions have the same denominator: $\frac{2\times 4}{5x\times 4}+\frac{3\times 5x}{4\times 5x}=0.5$ or $\frac{8}{20x}+\frac{15x}{20x}=0.5$. Next we add the two fractions to obtain $\frac{8+15x}{20x}=0.5$. Now we can multiply both the left and right hand side with $20x$ and rewrite as $8+15x=10x$. Finally we subtract $15x$ left and right of the $=$ sign and obtain $8=-5x$ or $x=-\frac{8}{5}$.

Example 2: solve the following equation: $\frac{2}{x+3}=5$ By multiplying left and right of the $=$ sign with $x+3$ we can rewrite this as $2=5(x+3)$ or $2=5x+15$. Next we move 15 to the other side of the $=$ sign by subtracting 15 left and right of the $=$ sign, resulting in $-13=5x$ and hence $x=\frac{-13}{5}$

So what about these two populations of rabbits that we started with? For population one we have to add the two fractions describing the two different causes of death to find out the overall fraction of rabbits dying in that population:

$$\frac{1}{2}+\frac{1}{5}=\frac{1\times 5}{2\times 5}+\frac{1\times 2}{5\times 2}=\frac{5}{10}+\frac{2}{10}=\frac{7}{10}$$

To compare in which populations more rabbits die we next need to compare this obtained fraction with the fraction of rabbits dying in the second population. For this we need to ensure that the fractions have the same denominator:

$$\frac{7}{10}=\frac{7\times 7}{7\times 10}=\frac{49}{70}$$

$$\frac{4}{7}=\frac{4\times 10}{7\times 10}=\frac{40}{70}$$

$$\frac{49}{70}>\frac{40}{70}$$

Thus in the first population a larger fraction of rabbits dies than in the second population, implying that the second population is doing better.

3.3 Exercises

1. Compute, and simplify if possible

   (a) $\frac{1}{7}\times\frac{2}{4}$
   (b) $\frac{1}{6}-\frac{2}{3}$
   (c) $\frac{2}{3}:\frac{4}{5}$
   (d) $\frac{4}{12}+\frac{5}{36}$

2. A family has an income of 2400 euro. $\frac{1}{3}$ of the income is spent on mortgage, $\frac{1}{7}$ on groceries and $\frac{1}{4}$ on heating, water, electricty and insurances. From the tax office they receive back $\frac{1}{3}$ of their mortgage payments. How much money is left at the end of the month? Compute this by summarizing all expenses into a single fraction.

3. Write as a single fraction:

$$\frac{6}{r}-\frac{5r}{30r+5}$$

4. Solve x from the following equation:

$$\frac{a}{b}x-\frac{c}{d}=\frac{e}{f}$$

5. Solve A from the following equation:

$$25A : \frac{3}{7} = 12 - \frac{1}{3}A$$

6. Solve N from the following equations:

   (a) $(b - \frac{N}{k})N = 0$
   (b) $(b - d(1 + \frac{N}{k}))N = 0, \;\; d \neq 0;\; k \neq 0$
   (c) $(\frac{b}{1+N/h} - d)N = 0,\; b \neq 0;$

Chapter 4

Algebraic equations

4.1 Introduction to Algebraic expressions

Algebraic expressions may contain numbers $(1,2,3,...)$, variables $(x,y,N,...)$, parameters $(a,b,c,...)$ and arithmetic operations $(+,-,\times,\div)$. Below, we review examples of several basic operations which help us to work with algebraic expressions.

One of the most basic algebraic operations is getting rid of parentheses (opening parentheses) to simplify the expression. For that we use the following rule:

$$(a+b)(c+d) = ac + ad + bc + bd$$

Note, that here ac means $a \cdot c$, etc., as the multiplication sign is often omitted.

Example (open parentheses): $(4x+2a)(2-3x) = 8x - 12x^2 + 4a - 6ax$

Note that while the left-hand side contains two bracketed terms that are multiplied with one another, the right-hand side contains four terms that are added to or subtracted from one another. Depending on the situation, either version could be more useful to work with. Therefor, we can also introduce parentheses to decompose an expression into multiple factors. For example, we may have:

$$9x^3 + 3x^2 - 6a^4x^2$$

Since all terms contain (at least) an x^2 term, we can write the expression as x^2 multiplied by all the terms divided by x^2:

$$9x^3 + 3x^2 - 6a^4x^2 = x^2(9x + 3 - 6a^4)$$

This manipulation commonly referred to as “factoring out” a term. In this case, we factored out x^2. We could furthermore also factor out the 3, by dividing each term within the brackets by 3 and putting it outside of the brackets:

$$x^2(9x + 3 - 6a^4) = 3x^2(3x + 1 - 2a^4)$$

4.2 Solving of equations

An equation is a mathematical relationship involving one or more unknown variables. Solving equations means finding the values of these unknowns. When these values are put back into the equation, the left- and right-hand sides of the equations should be equal. For example: the equation $2x-16=-10$ has as a solution $x=3$, because filling in $x=3$ yields $2\cdot 3-16=6-16=-10$.

Solving equations containing a single variable means finding one or more values for the variable for which the equation holds true. In order to find these solutions we need to move all terms containing the variable to one side of the $=$ sign. This can be achieved by multiplying the left and right hand side of the equation by the same amount, dividing both sides by the same amount, or by adding or substracting the same amount from the left and right hand side of the equation. Let's apply this to the equation above:

Starting with:	$2x-16=-10$
Adding 16 to both sides:	$2x=6$
Dividing by 2:	$x=3$

When an equation contains multiple instances of your variable, it may require some rewriting to solve, for example:

Starting with:	$2x^2+4x=0$	
Factoring out x:	$x(2x+4)=0$	Now we have an equation in terms of A $\times$ (B)
Equation holds if A is 0, so:	$x=0$	
But also if B is 0, so:	$2x+4=0\rightarrow x=2$	

Sometimes factoring out x like this is not possible, because there is a third (or fourth) term that does not contain x. This is the case for quadratic equations such as $ax^2+bx+c=0$. We can use the so-called 'abc' formula, which gives us the two possible solutions as $x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Example: solve a more complex equation with one variable: $aN^2=bN$
First bring all terms with N to one side, so $aN^2-bN=0$, than factor out N, so $N(aN-b)=0$. Now we once again have a form of $A\times B=0$, meaning that if either $A=0$ or $B=0$, the equation holds. So, for our equation $N=0$ or $aN-b=0$. So finally, $N=0$ or $N=\frac{b}{a}$.

If an equation contains multiple variables, you can choose which variable you will put on its own to one side of the $=$ sign. You then solve the equation and express the solution in terms of this particular variable. Which variable you choose for this is up to you, so it is best to pick the variable for which solving the equation is easiest. The solution of the equation then describes the value of this variable as a function of the values of the other variables that were in the equation. Try this with the following examples:

Example: solve the equation for one of the two variables: $aN-bM^2=0$
Note that N is simpler to solve for than M, so $aN=bM^2$ and hence $N=\frac{b}{a}M^2$.

Example: solve the equation for one of the two variables: $aN=b\frac{M}{N}$
Note that M only appears once, while N appears twice. M is probably easier to solve for than N. So, let's first multiply left and right with N. This give us $aN^2=bM$ and hence $M=\frac{a}{b}N^2$.

Finally, note that many complex equations can not be solved analytically. As an example, higher order equations, also called polynomials, of the form $a+bx+cx^2+dx^3+\cdots=0$ can only be solved if we can factor them into multiple terms of lower order that then each can be solved separately. Sometimes another option is to introduce a new variable that allows you to rewrite your equation into one that is easily solved:

Example: solve the following equation: $aN^6 - bN^3 = 0$.
Note that this can be rewritten as $a(N^3)^2 - bN^3 = 0$, so both terms can be written in terms of N^3. We can therefore introduce $M = N^3$ and rewrite the equation as $aM^2 - bM = 0$. We already know how to solve this from earlier examples: $M(aM - b) = 0$ and hence $M = 0$ or $aM - b = 0 \rightarrow M = \frac{b}{a}$. Therefore $N^3 = 0$ and thus $N = 0$, or $N^3 = \frac{b}{a}$ and thus $N = (\frac{b}{a})^{1/3}$.

4.3 Solving systems of equations

Sometimes we are working with multiple algebraic equations which contain the same variables. Such systems are called “coupled algebraic equations”, and solving them is a little bit more work. However, in principle, it works the same as solving several independent equations. An important additional step, however, is to substitute solutions of one equation into the other, such that we can find the solution that applies to both equations.

As an example, consider the equations $3x + 2y = 0$ and $y/5 = 3$. If these equations form a system (are coupled), we are now looking for a *combination* of x and y for which both equations hold. Solving the first equation gives you $3x = -2y$ and hence $x = -(2/3)y$. Solving the second equation is very easy, and gives you $y = 15$. Note that the first solution is expressed in terms of an unknown y (namely, minus two-thirds of this unknown y), while the second equations simply yields a value for y. So we can fill in this value of $y = 15$ into the first solution, to get rid of this variable: $x = -(2/3)y = -(2/3) \cdot 15 = -10$. Hence, we now know that when $x = -10$ and $y = 15$, both equations are true.

Note that if you are solving a system of two or more algebraic equations it is not necessarily the best approach to start with the first equation and work your way down. Instead, the smart approach is to start with the equations that are most easy to solve. Indeed, in the example the second equation is much easier than the first, and we used it’s solution to find the final solution of the first equation. Furthermore, only fill in a solution of one equation into a solution of another equation if this actually simplifies matters.

Example 1: solve the following system of equations: $0 = aN - bM$ and $0 = M(cN - d)$
From the first equation we obtain $N = \frac{b}{a}M$. From the second equation we obtain $M = 0$ or $cN - d = 0$ and hence $N = \frac{d}{c}$. If we now combine this with the solution of the first equation, $M = 0$ gives us $N = 0$ and $N = \frac{d}{c}$ gives us $\frac{d}{c} = \frac{b}{a}M$ and hence $M = \frac{da}{cb}$. So we have two solutions $N = 0$, $M = 0$ and $N = \frac{d}{c}$, $M = \frac{da}{cb}$.

Example 2: solve the following system of equations for n and p: $\begin{cases} an - an^2 - bnp = 0 \\ np - kp = 0 \end{cases}$

Let us start with the second equation, as it looks easier. $np - kp = 0$ gives us $p(n - k) = 0$ and hence $p = 0$ or $n = k$. Now let us move back to the first equation, $an - an^2 - bnp = 0$. First, we fill in $p = 0$. This gives us $an - an^2 = 0$ and hence $an(1 - n) = 0$ and hence $n = 0$ or $n = 1$. This means that from filling in only the first solution of the first equation we already obtained two overall solutions for this system $n = 0$,$p = 0$, and $n = 1$, $p = 0$. Let us also fill in the second solution of the second equation ($n = k$) into the first equation, this gives us $ak - ak^2 - bkp = 0$. Note that as k is a parameter, $k = 0$ is not a solution!. However, we can divide all by k, resulting in $a - ak - bp = 0$ or $a - ak = bp$ and hence

$p = \frac{a-ak}{b}$. The third overall solution thus is $n = k$, $p = \frac{a-ak}{b}$. Summarizing, we found the following (n, p) values as possible solutions to the given system of equations: $(0,0)$, $(1,0)$, $(k, \frac{a-ak}{b})$.

4.4 Exercises

1. Simplify the following expression:

   (a) $(ax-2by)(3y-4bx)+2b(2ax^2+3y^2)-8xyb^2$

2. Rewrite this equation so it can be solved with the abc-formula:

   (a) $\frac{6}{r} = \frac{5r}{30r+5}$

3. Solve the equation for the specified variable:

   (a) find r in: $3r+2-5(r+1) = 6r+4$
   (b) find x in: $x+\frac{4}{x} = 4$
   (c) find N in: $(b-\frac{N}{k})N = 0$
   (d) find N in: $(b-d(1+\frac{N}{k}))N = 0,\ d \neq 0;\ k \neq 0$
   (e) find N in: $(\frac{b}{1+N/h}-d)N = 0,\ b \neq 0;$

4. Solve the system of equations for the specified variables:

   (a) find x, y in: $\begin{cases} x-2y = -5 \\ 2x+y = 10 \end{cases}$
   (b) find x, y in: $\begin{cases} ax+by = 0 \\ cx+dy = -b \end{cases}$
   (c) find x, y in: $\begin{cases} x(1-2x)+xy = 0 \\ 4y-xy = 0 \end{cases}$
   (d) find x, y in: $\begin{cases} 4x-xy-x^2 = 0 \\ 9y-3xy-y^2 = 0 \end{cases}$
   (e) find R, N in: $\begin{cases} b(1-\frac{R}{k}-d-aN)R = 0 \\ (R-\delta)N = 0 \end{cases}$, $a,b,d,k,\delta \neq 0;$

Chapter 5

Limits and Asymptotes

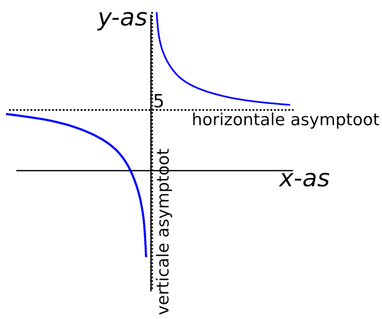
5.1 Introduction

It is often important and interesting to know the long term behaviour of biological systems. Think for example of a growing population, for which we would like to know whether in the long run its size stabilizes at a certain number of individuals or that it keeps perpetually increasing. If we describe the dynamics of a biological system as $f(x)$ than we can derive the long term behaviour of the system by filling in $x = \infty$. The value of the function $f(x)$ for x approaching ∞ is called the limit of the function. We speak of a limit since x can go closer and closer to ∞ but can never actually reach it. The value of $f(x)$ for x approaching ∞ is also referred to as the asymptotic behaviour of $f(x)$.

To be able to qualitatively sketch the graph of a function, it is important to know the behaviour of the function for x approaching $+\infty$ and for x approaching $-\infty$. As an example consider $f(x) = 5 + \frac{1}{x}$. For $x \to \pm\infty$ (x going to $\pm\infty$) $f(x)$ approaches (but never quite reaches) 5. If the asymptotic behaviour of a function is a constant value (that is x goes to a value a rather than going to $\pm\infty$) this is called an asymptot of the function. For $f(x) = 5 + \frac{1}{x}$ we obtain for $x \to \pm\infty$ the asymptote $y = 5$. Since it is a constant value for y and hence corresponds to a horizontal line, we call it a horizontal asymptote. Now as an alternative example consider $g(x) = 3x^2$. For $x \to \pm\infty$ $g(x)$ will also go to $+\infty$. Thus the asymptotic behaviour of this function is that it keeps quadratically increasing, and hence there is no asymptotic line that the function will approach.

In addition to the limit for $x \to \pm\infty$, for functions with fractions it is important to consider what happens when the denominator of the fraction is almost zero. Let us again consider the function $f(x) = y = 5 + \frac{1}{x}$. For $x = 0$ the function is not defined since we can not divide by zero. However, in order to be able to understand the behaviour of the function and sketch a graph of this function we need to know the behaviour of $f(x)$ for x close to 0. For x slightly larger than zero we obtain a division by a very small positive number, which gives a very large positive outcome. Thus for $x \downarrow 0$ (x approaches zero from "above") which can also be written as $x \to 0+$ (x approaches zero from numbers larger than zero) $f(x)$ approaches $+\infty$. In contrast, for x slightly smaller than zero we obtain a division by a very small negative number , which results in a very large negative outcome. Thus for $x \uparrow 0$ (x approaches zero from "below") alternatively written as $x \to 0-$,(x approaches zero from numbers smaller than zero) $f(x)$ approaches $-\infty$. Since in this case the line that is approached by the function $f(x)$ is $x = 0$ which is a vertical line, we call this a vertical asymptote of the function.

Thus, we know that $f(x)$ approaches the horizontal asymptote $y = 5$ both for $x \to +\infty$ and for $x \to -\infty$, and we know that $f(x)$ goes to $+\infty$ when approaching the vertical asymptote $x = 0$ from the right and to $-\infty$ when approaching it from the left. This now allows us to sketch $f(x)$:



5.2 Rules for working with limits

We call A the limit of the function $f(x)$ when for x approaching a the value of $f(x)$ approaches A. Mathematically this is written as:

$$\lim_{x \to a} f(x) = A$$

A lot of limits can be obtained trivially by filling in $x = a$ into $f(x)$:

$$\lim_{x \to a} f(x) = f(a)$$

Example:

$$\lim_{x \to 5} 3x^2 + 7$$

Let us fill in $x = 5$ into $f(x) = 3x^2 + 7$: $f(5) = 3 \cdot 5^2 + 7 = 3 \cdot 25 + 7 = 75 + 7 = 82$. So $\lim_{x \to 5} 3x^2 + 7 = 77$.

There are two important exceptions. The first exception arises when we wish to find the limit of $f(x)$ for x going to $\pm\infty$. As described in the introduction these types of limits are important for understanding the behaviour of $f(x)$ and for drawing the graph of $f(x)$. However, this limit can not be simply computed by filling in $x = \infty$ as ∞ is not a concrete value that we can do computations with.

One possible approach is to fill in very large values for x and study the behaviour of $f(x)$. Thus, for $f(x) = 3x^2 - 5$ we will find that for very large x $f(x)$ also becomes very large and hence that $\lim_{x \to +\infty} f(x) = +\infty$. However, for functions $f(x)$ containing parameters rather than constant values, this approach is not generally applicable. Consider for example $f(x) = ax^3 - bx^2$. Filling in large numbers for x results in large positive values for the term ax^3 and large negative values forthe term $-bx^2$. However, since we do not know a and b we can not compute which term dominates. In these cases we will need to apply a different technique, taking into consideration which term contains the highest power of x. In this example ax^3 has a power 3 and $-bx^2$ a power 2, thus the first term increases considerably faster with increasing x than the second. Thus, the first term dominates and $f(x)$ goes to $+\infty$ for x going to $+\infty$.

This problem is particularly relevant for so-called rational functions, which take on the form $f(x) = \frac{p(x)}{g(x)}$, in which both $p(x)$ and $g(x)$ contain powers of x. In these situations we can find the limit by using the following two properties of power functions:

$$\lim_{x\to\infty} \frac{C}{x^{\alpha}} = 0$$

$$\lim_{x\to\infty} \frac{x^{\alpha}}{C} = \infty$$

with $C > 0$ and $\alpha > 0$

The logic behind the first property is that if x goes to ∞ (and thus becomes bigger and bigger) the term x^{α} ($\alpha > 0$) also becomes bigger and bigger, and hence the division $\frac{C}{x^{\alpha}}$ becomes smaller and smaller, approaching zero.

The logic behind the second property is the reverse: as x goest to ∞ (and thus becomes bigger and bigger) the term x^{α} ($\alpha > 0$) also becomes bigger and bigger and hence the division $\frac{x^{\alpha}}{C}$ becomes larger and larger, approaching infinity.

In order to use these rules to compute a limit we need to do the following: (1) determine what is the highest power of x in our function $\frac{p(x)}{g(x)}$, (2) divide each term in our function by this highest power of x, and (3) compute the limit for each individual term using the rules $\lim_{x\to\infty} \frac{C}{x^{\alpha}} = 0$, en $\lim_{x\to\infty} \frac{x^{\alpha}}{C} = \infty$, and finally (4) find the overall limit from these individual limits.

Example:

$$\lim_{N\to\infty} \frac{aN^2 - 3N}{3 - 2N^2}$$

The highest power here is N^2. Dividing by N^2 gives $\frac{\frac{aN^2}{N^2} - 3\frac{N}{N^2}}{\frac{3}{N^2} - \frac{2N^2}{N^2}} = \frac{a - \frac{3}{N}}{\frac{3}{N^2} - 2}$. Thus the limit is: $\frac{a-0}{0-2} = -\frac{a}{2}$

Example:

$$\lim_{x\to\infty} \frac{ax^3 - bx^2 + c}{ax^4 - b}, a, b, c > 0$$

Using the same approach as above we obtain: $\frac{ax^3 - bx^2 + c}{ax^4 - b} = \frac{\frac{ax^3}{x^4} - \frac{bx^2}{x^4} + \frac{c}{x^4}}{\frac{ax^4}{x^4} - \frac{b}{x^4}} = \frac{\frac{a}{x} - \frac{b}{x^2} + \frac{c}{x^4}}{a - \frac{b}{x^4}} = \frac{0-0+0}{a-0} = \frac{0}{a} = 0.$

Example:

$$\lim_{P\to\infty} \frac{aP - 3bP^3}{cP^2}, a, b, c > 0$$

Applying again the same approach we get: $\frac{aP-3bP^3}{cP^2} = \frac{\frac{aP}{P^3} - \frac{3bP^3}{P^3}}{\frac{cP^2}{P^3}} = \frac{\frac{a}{P^2} - 3b}{\frac{c}{P}} = \frac{0-3b}{0} = -\infty$. Note that the one but last expression seems to result in a division by zero. However, by considering the separate elements we see that $\frac{c}{P}$ is a very small positive number.$-3b$ divided by a very small positive number then gives us a limit of $-\infty$.

The second exception to simply filling in $x = a$ arises when $x = a$ results in a division by zero. An example is $\lim_{x\to 3} \frac{2}{x-3}$. In this case we cannot fill in $x = 3$ as this results in a division by zero. As for the limits of x going to plus or minus infinity, these limits require a more detailed analysis.

Using our calculator we can fill in values for x that lie closer and closer to 3. When we approach $x = 3$ from the right (> 3) and fill in a series of values like $x = 3.1; 3.05; 3.01, 3.005;$, the value of $f(x)$ increases further and further, while if we approach $x = 3$ from the left (< 3),and fill in a series of values like $x = 2.9; 2.95; 2.99, 2.995;$ $f(x)$ obtains larger and larger negative values. This we can write as $\lim_{x\downarrow 3} \frac{2}{x-3} = +\infty$,and $\lim_{x\uparrow 3} \frac{2}{x-3} = -\infty$ (or alternatively $\lim_{x\to 3+} \frac{2}{x-3} = +\infty$,and $\lim_{x\to 3-} \frac{2}{x-3} = -\infty$).

Again, if we have a function with parameters rather than constant values we can not fill in numbers. Consider for example $lim_{x\to a}\frac{x}{x-a}$. In this case we have to consider whether we are dividing by a small positive or negative numbers, and hence whether the outcome will be a large positive or negative number. Thus $lim_{x\downarrow a}\frac{x}{x-a} = +\infty$ and $lim_{x\uparrow a}\frac{x}{x-a} = -\infty$.

5.3 Exercises

Calculate the following limits:

$$\lim_{x\to\infty} 3x^2 - 2x + 5$$

$$\lim_{x\to 0+} \frac{1}{x}$$

$$\lim_{x\to 0-} \frac{1}{x}$$

$$\lim_{x\to -1} \frac{(2+2x)}{(1+x)}$$

$$\lim_{x\to\infty} \frac{(x^2+3)}{x^3}$$

$$\lim_{x\to\infty} \frac{ax^2}{(bx^2+x)}$$

$$\lim_{x\to\infty} \frac{(ax^2+b)}{cx^3} + d$$

Sketch a graph of the following functions and their asymptotes

$$f(x) = \frac{x}{x-2} + 1$$

$$f(x) = \frac{ax}{bx+c} - d$$

, with $d > a/b$

Chapter 6

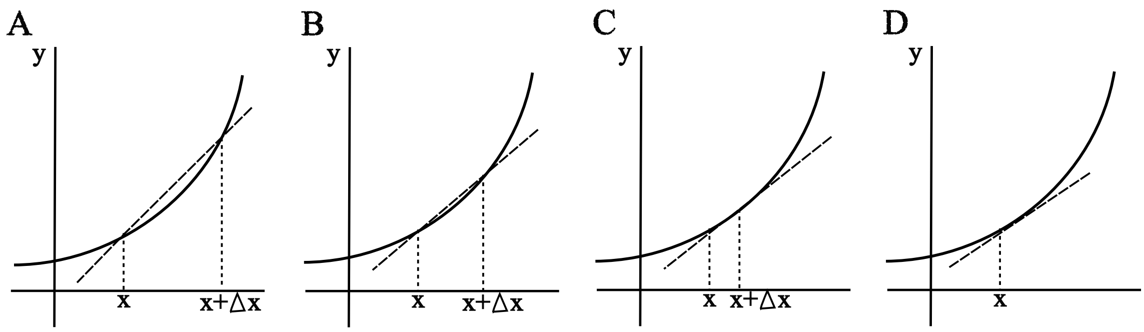
Derivatives

6.1 Introduction

What is a derivative of a function, and what is its use? Let us consider this graphically. In Figure 1 we draw several times the same simple, increasing function $f(x)$ combined with a straight line intersecting this function in two points x and $x+\Delta x$. Going from Figure 1A to 1D the distance between the points x and $x+\Delta x$ is gradually decreasing. For the slope (*richtingscoefficient* or *hellingshoek* in Dutch) of these straight lines we can write the following equation:

$$\frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

This simply says that the slope equals the distance travelled along the y-axis divided by the distance traveled along the x-axis.



We can see that as we go from Figure 1A to 1D and the distance between x and $x+\Delta x$ becomes smaller and smaller the slope of the straight intersection line more and more approaches the (local) slope of the function it is intersecting. When the distance between the two intersection points goes to zero and the points x and $x+\Delta x$ coincide the slope of the straight line equals the (local) slope of the function. Note that the straight line now no longer intersects with but is tangential to the function (*raaklijn* in Dutch). Based on these observations we can write the following equation for the slope of a function in a point x:

$$lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{df}{dx}$$

The slope $\frac{df}{dx}=f'(x)$ is called the derivative of function $f(x)$.

It follows from the above graphical and mathematical definitions that the derivative describes the rate of change in the value of $f(x)$ as a function of a small change in the value of x. Imagine for example that in Figure 1D the function $f(x)$ describes the number of rabbits in a population and x describes time in months. De derivative $f'(x)$ then describes the rate at which the number of rabbits changes at time point x. Put differently, by considering not only the value of $f(x)$ but also the value of its derivative $f'(x)$ we know of how many rabbits the population consists at time x but we also know of how many rabbits the population is expected to consist at $x+\Delta x$, as we know its rate of change.

6.2 Rules for finding derivatives

Below the general rules for finding derivatives of functions are provided:

Powerfunctions (*machtsfuncties*):
$f(x) = ax^n \rightarrow f'(x) = (a \times n)x^{n-1}$
$f(x) = \frac{b}{x^n} \rightarrow f'(x) = \frac{b \times -n}{x^{n+1}}$

Chainrule (*kettingregel*):
$F(x) = f(g(x)) \rightarrow F'(x) = f'(g(x)) \times g'(x)$

Productrule (*productregel*):
$F(x) = f(x) \times g(x) \rightarrow F'(x) = f'(x) \times g(x) + f(x) \times g'(x)$

Quotientrule (*quotientregel*):
$F(x) = \frac{f(x)}{g(x)} \rightarrow F'(x) = \frac{f'(x) \times g(x) - f(x) \times g'(x)}{g^2(x)}$

Special cases:
$f(x) = e^x \rightarrow f'(x) = e^x$
$f(x) = \ln(x) \rightarrow f'(x) = \frac{1}{x}$
$f(x) = a^x \rightarrow f'(x) = \ln(a)a^x$
$f(x) = log_a(x) \rightarrow f'(x) = \frac{1}{x\ln(a)}$
$f(x) = sin(x) \rightarrow f'(x) = cos(x)$
$f(x) = cos(x) \rightarrow f'(x) = -sin(x)$
$f(x) = tan(x) \rightarrow f'(x) = \frac{1}{cos^2(x)}$

6.3 Exercises

1. Compute the derivatives of the following functions:

   (a) $f(x) = x^2 + 15x$
   (b) $f(x) = 12x^5 - 6$

(c) $f(x) = \frac{7}{x^4}$
(d) $f(x) = ax^3 - bx^5$

2. Compute the derivatives of the following functions:

(a) $f(x) = 3e^{5x^2}$
(b) $f(x) = 12x^3 \times 6x^4$
(c) $f(x) = cos(ax - bx^3)$
(d) $f(x) = \frac{5x+6x^2}{2x}$

3. Compute the derivatives of the following functions:

(a) $f(x) = 3\ln(4x^3)$
(b) $f(x) = sin(ax)cos(ax)$
(c) $f(x) = \frac{3x^2}{6x}$
(d) $f(x) = sin^2(x) + cos^2(x)$
(e) $f(x) = (a - x) \cdot (b + x) \cdot (c - x)$

Chapter 7

Drawing Functions

7.1 Introduction

Functions describe the relationship between variables. For example, $y = f(x)$ may describe how much a person on average weighs (with y representing weight) as a function of his or her height (with x representing height) or it may describe the number of bacteria in a petridish (y) as a function of time since the experiment started (x). Functions thus represent an important means to describe and summarize biological data.

To obtain insight in the relationship that a function $f(x)$ describes it is important for you to be capable of sketching the shape of the function. For this we will need to determine properties of the function such as intersection points with the x and y axes, whether or not the function has asymptots, and possibly the location of maxima or minima. To determine these properties we will need the techniques described in earlier chapters of this appendix on solving equations, finding limits and determining derivatives.

7.2 Common functions

Functions are most frequently represented by drawing their graph. To do this we plot the value of the variable x on the x axis and the value of the function $f(x)$ on the y axis. Below we show a series of frequently occuring graphs.

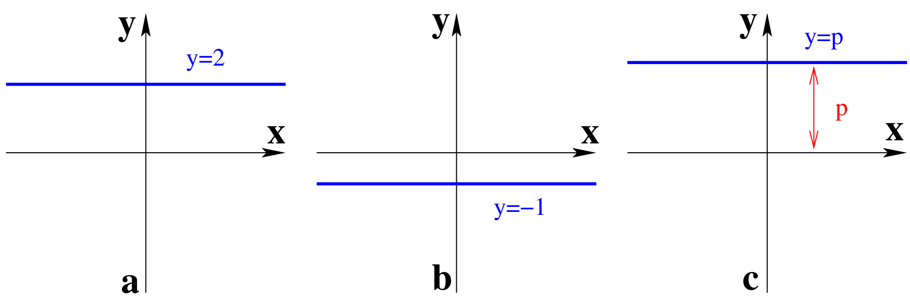


Figure 7.1: *"Zero order" graphs 1* The graph of a function of the form $y = p$ is a horizontal line intersecting the y axis at the value p (fig.7.1).

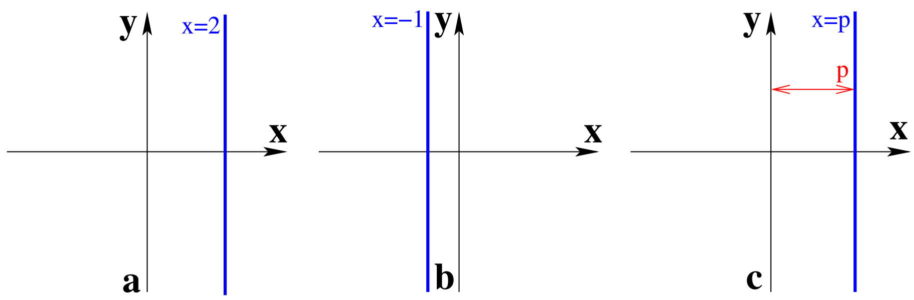


Figure 7.2: *"Zero order" graphs 2* The graph of a function of the form $x = p$ is a vertical line intersecting the x axis at the value p.

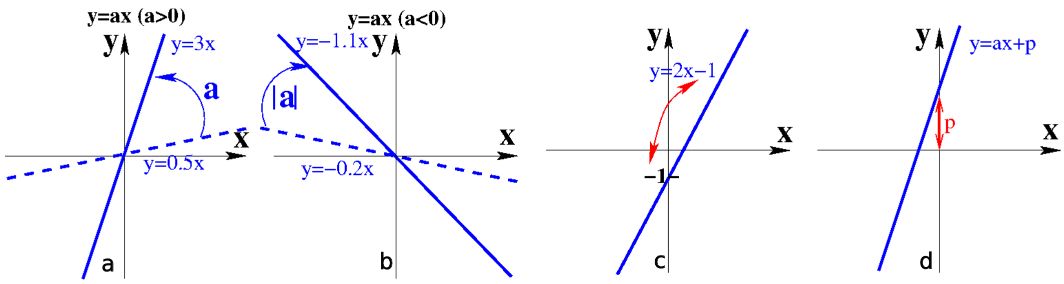


Figure 7.3: *Linear (first order) graphs* The graph of the linear function $y = ax + p$ is a straight line with a slope a and intersection point with the y axis p. The larger the absolute value of a the steeper the slope of the line. For $a > 0$ the slope is positive, resulting in an increasing line, while for $a < 0$ the slope is negative, resulting in a decreasing line. p determines the shift of the graph relative to the x axis, for $p > 0$ the graph is shifted upward having an intersection point with the positive y axis, while for $p < 0$ the graph is shifted downward, having an intersection point with the negative y axis.

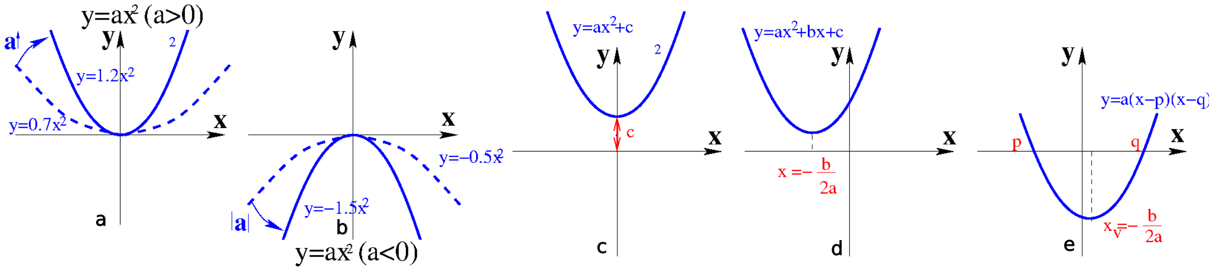


Figure 7.4: *Quadratic (second order) graphs* The graph of the quadratic equation $y = ax^2 + bx + c$ describes a parabola. If $a > 0$ the parabola has a minimum with its two ends going upwards (a so-called *dalparabool*), while if $a < 0$ it has a maximum with its two ends going downwards (a so-called *bergparabool*). The larger the absolute value of a the steeper the parabola increases. The parameter c determines the vertical shift relative to the x axis and determines whether the parabola intersects with the y axis. The parameter b determines the horizontal shift relative to the y axis. We can compute that this horizontal shift is given by $-\frac{b}{2a}$. This can be done by finding the maximum or minimum of the function. At this point the derivative of the function should equal zero. So if $f(x) = ax^2 + bx + c$ then $f'(x) = 2ax + b$. Solving $f'(x) = 2ax + b = 0$ then gives $x = -\frac{b}{2a}$ which is the x coordinate of the maximum or minimum of the function, and hence the shift of this maximum or minimum relative to the y axis. A parabola has a maximum of two intersection points with the x axis. These intersection points can be found by solving $f(x) = ax^2 + bx + c = 0$ using the abc-formula: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. If the **discrimant** of the abc-formula $D = b^2 - 4ac$ is > 0 the abc-formula has 2 solutions and hence the parabola has 2 intersections with the x axis with the maximum or minimum of the parabola lying exactly in the middle of these two intersection points. If instead $D = 0$ the abc-formula has only 1 solution and the parabola is tangential to the x axis at this point. This point then corresponds to the maximum or minimum of the parabola. Finally, if $D < 0$ the abc-formula has no solution, and hence the parabola has no point in common with the x axis. Thus the parabola lies entirely above or below the x axis.

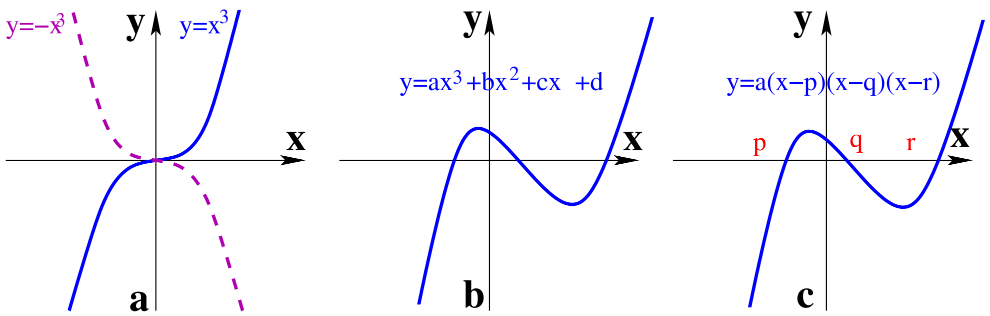


Figure 7.5: *Cubic (third order) graphs* For a general cubic function $y = ax^3 + bx^2 + cx + d$ we have many more possibilities than for a quadratic function, which we will not all discuss here. The two simplest forms are given by the functions $y = x^3$ and $y = -x^3$ depicted in Fig.7.5a. Important here is the asymptotic behavior of the function at $x \to \pm\infty$. For $y = x^3$ we see that y goes to $+\infty$ when x increases and to $-\infty$ when x decreases; for $y = -x^3$ we have exactly the opposite situation. This asymptotic behaviour also holds for graphs of the general cubic function $y = ax^3 + bx^2 + cx + d$: it is the ax^3 term (whether it is postive or negative) that determines the asymptotic behaviour. While the graphs $y = x^3$ and $y = -x^3$ only have a single intersection point with the x axis, a graph of the general function $y = ax^3 + bx^2 + cx + d$ may have up to three intersection points that can be found from solving the equation $ax^3 + bx^2 + cx + d = 0$. In addition, while the graphs $y = x^3$ and $y = -x^3$ only have a bending point, a graph of the general function $y = ax^3 + bx^2 + cx + d$ may have up to two extrema: one maximum and one minimum (fig.7.5b). The extrema are points where the derivative of the function is zero, which in this case results in the following quadratic equation: $(ax^3 + bx^2 + cx + d)' = 3ax^2 + 2bx + c = 0$. If the intersection points with the x axis are known ($x = p, x = q, x = r$) and the asymptotic behavior is also known the function can be rewritten as $y = a(x-p)(x-q)(x-r)$ and the graph can easily be drawn as shown in Fig.7.5c.

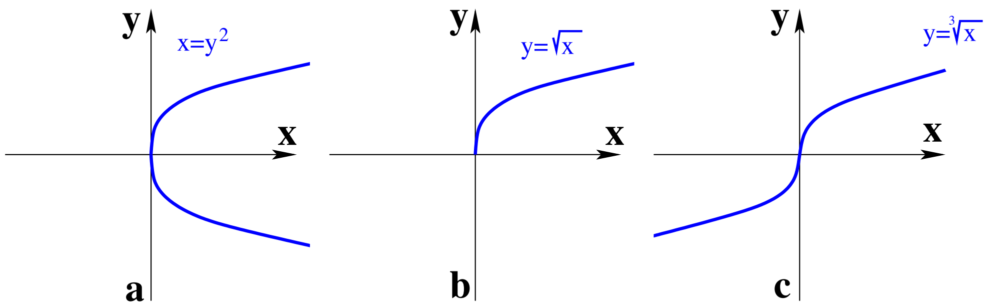


Figure 7.6: *Graphs of power functions* We show three examples of graphs of the power function x^a involving $0 < a < 1$ in Fig.7.6. As $0 < a < 1$ the function increases slower than the function $y = x$ and is concave downward (in the first quadrant). To draw graph $y = \sqrt{x}$ let us first draw function $x = y^2$. If in function $x = y^2$ we switch the x and y we will get $y = x^2$. The graph of $x = y^2$ can then be found by switching the x and the y-axis for the graph of the parabola $y = x^2$ in Fig.7.4a and we get the curve depicted in fig.7.6a. Now we need to consider that $x = y^2$ is equivalent to $y = \pm\sqrt{x}$. Thus, the upper branch in fig.7.6b corresponds to $y = \sqrt{x}$, whereas the lower branch corresponds to $y = -\sqrt{x}$. Thus, we need to only draw the upper arm of $x = y^2$ to obtain $y = \sqrt{x}$. Similarly, the graph of the function $y = \sqrt[3]{x}$ (Fig.7.6c) can be found by a 90^o rotation of the graph of the function $y = x^3$ from Fig.7.5a.

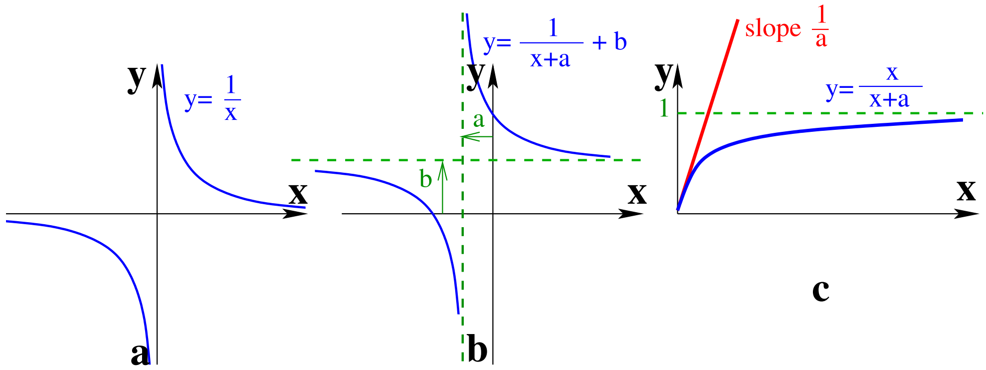


Figure 7.7: *Graphs of rational or hyperbolic functions 1* We show examples of graphs of rational or hyperbolic functions of the type $\frac{p(x)}{q(x)}$. The graph of the function $y = \frac{1}{x}$ (Fig.7.7a) has a vertical asymptote ($x = 0$) and a horizontal asymptote ($y = 0$). The graph of the function $y = \frac{1}{x+a} + b$ can be simply drawn by shifting the graph of the function $y = \frac{1}{x}$ with b in the y (vertical) direction and with $-a$ in the x (horizontal) direction. In this case the vertical asymptote (the value at which x goes to $\pm\infty$) lies at $x = -a$, since at this point the denominator in $\frac{1}{x+a}$ equals zero. The horizontal asymptote of the function $y = \frac{1}{x+a} + b$ has moved to $y = b$ since $\lim_{x\to\infty} \frac{x}{x+a} + b = b$. The functions $y = \frac{1}{x}$ and $y = \frac{1}{x+a} + b$ are called hyperbolic functions or hyperbola. Another well known rational function is the classical Michealis-Menten function describing saturation of y at increasing x: $y = \frac{x}{x+a}$. Fig.7.7c shows the graph of this function. Considering that in biological applications x and a can not take on negative values ($x \geq 0, a > 0$), we show this graph only for positive x and y values (first quadrant). Independent of the value of a the horizontal asymptote always lies at $y = 1$, since $\lim x \to \infty \frac{x}{x+a} = 1$. The slope of the function at $x = 0$ can be found from the derivative of the function $f'(x) = (\frac{x}{x+a})' = \frac{a}{(x+a)^2}$ at $x = 0$, which equals $f'(0) = \frac{1}{a}$.

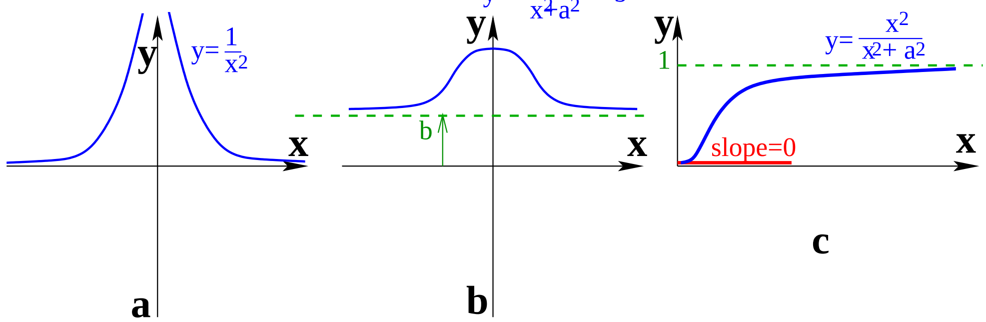


Figure 7.8: *Graphs of rational or hyperbolic functions 2* In Figure 7.8a,b we show similar functions as in Figure 7.7 but now for x to the power 2: $y = \frac{1}{x^2}$, $y = \frac{1}{x^2+a^2} + b$ and $y = \frac{x^2}{x^2+a^2}$. We see that the function $y = \frac{1}{x^2}$ has a graph that closely resembles the graph of the function $y = \frac{1}{x}$ for positive x values, while it has reversed y values for negative x values. The reason for this reversal is that in this case y values remain positive as x^2 always has a positive value. An important difference for the function $y = \frac{1}{x^2+a^2} + b$ relative to the function $y = \frac{1}{x+a} + b$ is that there is no longer a vertical asymptote for $x = -a$, because x^2 is always positive and hence can never be equal to $-a^2$ to result in a zero denominator. Instead of an asymptote it now has a maximum at $x = 0$. The function $y = \frac{x^2}{x^2+a^2}$ (and its more general version $y = \frac{x^n}{x^n+a^n}$) are often used in biology (for example for the growth of a population to the carrying capacity of the ecosystem) and is called a Hill-function. Similar to the graph of the function $y = \frac{x}{x+a}$ the graph of this function has a horizontal asymptote at $y = 1$ (Fig.7.8c). However, for small values of x the graph of $y = \frac{x^2}{x^2+a^2}$ increases less fast than that of $y = \frac{x}{x+a}$: the slope of the tangential line at $x = 0$ equals 0 instead of $1/a$, which you can find by computing the derivative of the function in the point $x = 0$.

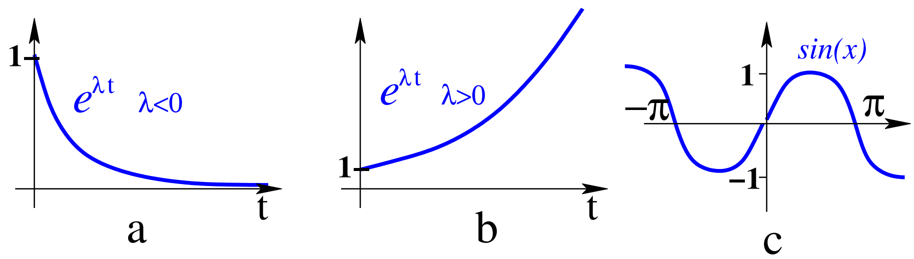


Figure 7.9: *Graphs of exponential and trigoniometric functions* Finally, in Fig.7.9 we show graphs of two functions that are important for systemsbiology: $e^{\lambda t}$ and $\sin(x)$. For growing values of t the function $e^{\lambda t}$ goes to zero if $\lambda < 0$ while the function goes to $+\infty$ if $\lambda > 0$. The function $sin(x)$ oscillates with a period of 2π between -1 and $+1$.

7.3 Rules for drawing functions

In this course it will often be necessary to draw the graph of f which is a function of a variable x but also depends on one or more constant valued parameters a, b, c for which the values are not a proiri known. As we saw above in the description of common functions, the value of a parameter may have a significant influence on the shape of a graph, for example whether a parabola has a maximum or minimum or whether a straight line intersects the y axis at positive or negative values. If we do not know the value of the parameter we can not determine which of these situations apply and hence we should consider (and draw) all possible situations.

How should you draw these functions with parameters? There are three complementary approaches possible.

The first possibility is to simplify the equation of the function and try to find to which known class of functions (linear, quadratic, cubic, exponential, etcetera) it belongs. If you know the graphs corresponding to these general functions you can derive how the graph for the function you have been given should look. For example, the graph of the function $y = f(x) + p$ can be obtained by a vertical shift by p units of the graph of $y = f(x)$ (assuming that this is a graph you know well). Similarly the graph of the function $y = (x-a)^2$ can be obtained by a horizontal shift of a units of the graph of $y = x^2$. Remember to determine whether changes in general graph shape occur if parameter values change.

The second possibility is to analyse the function and determine whether and where it intersects the x-axis and y-axis, whether and where it has asymptotes, and whether and where it has maxima:

1. Find intersection points with the x axis by solving $y = f(x) = 0$ for x.

2. Find intersection points with the y axis by filling in $x = 0$ in $y = f(x)$ and finding the value of y

Note that not all functions have intersection points with the x axis or y axis.Thus, if $f(x) = 0$ has no solution (for example in the abc formula $D < 0$, see earlier) there is no intersection point with the x axis, while if filling in $x = 0$ produces no constant value for y there is no intersection point with the y axis.

3. Find the horizontal asymptots by computing the limits $y = \lim x \to +\infty$ and $y = \lim x \to -\infty$. If the value of these limits is a constant number the function has a horizontal asymptote at this value. If instead the value of these limits is $\pm\infty$ the function has no horizontal asymptote. However, we did obtain usefull information on the behaviour of the graph of this function.

4. For rational functions of the form $f(x) = \frac{p(x)}{q(x)}$, find vertical asymptotes by determining whether there are x values for which $q(x) = 0$ and hence $f(x) \to \pm\infty$. These x values then are vertical asymptotes of the function.

Note that not all functions contain horizontal or vertical asymptotes.

5. Locations (x-value) of maxima and minima of the function $f(x)$ can be found by determining the derivative $f'(x)$ and determining for which x values $f'(x) = 0$. By subsequently filling in these x values in $f(x)$ you can find the height (y-value) of the maxima and minima.

6. If you need to know whether the function increases or decreases for a certain range of x values (for example between two intersection points with the x-axis), you can determine the value of the derivative

$f'(x)$ for these x values. $f'(x) > 0$ means that $f(x)$ locally increases as a function of x, $f'(x) < 0$ means that $f(x)$ locally decreases as a function of x.

The above rules can be summarized in the following figure: fig.7.10.

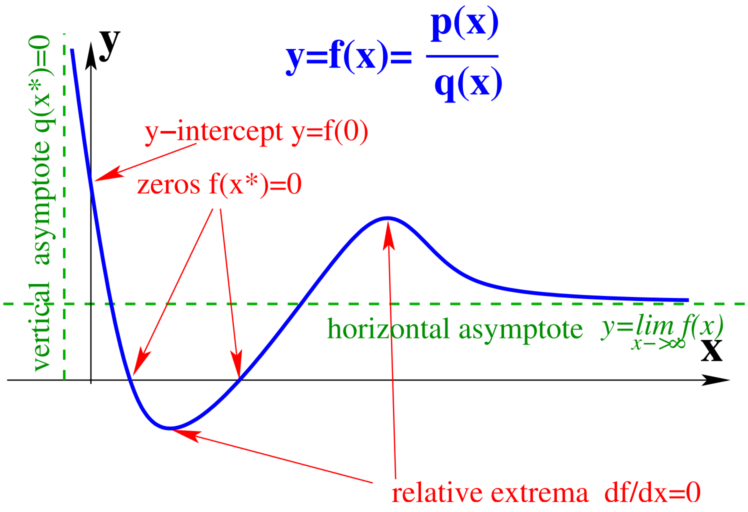


Figure 7.10:

For all the above aspects: intersection points, asymptotes, maxima and minima and slopes hold that they may depend on parameter values. Thus, if parameter values are not explicitly given, different function shapes may be possible and you need to draw the different possibilities.

The third and final possibility is to use your graphical calculator to draw the function by filling in "reasonable" values for the parameters. Remember to always try out multiple combinations of parameter values to make sure you find all possible graph shapes. Probably, to you this final possibility appears most easy and attractive. Note however that with this last approach, without having some basic understanding of the shape of general functions and how this shape and major shape determinants such as intersection points, asymptotes and maxima and minima may change due to parameter changes, it is very hard to determine when you have found all possible function shapes and can stop trying out different combinations of parameter values!

In summary, the second, analytical approach to drawing functions is the most general and hence preferred approach. For this approach you do not need to know by heart a lot of function shapes, as for the first approach. Also, you will automatically find whether intersection points, asymptotes or maxima depend on parameters and hence do not risk to overseeing certain possibilities, as for the third approach.

7.4 Exercises

For the following, determine (if present) 1) intersection points with the x and y axes 2) horizontal and vertical asymptotes and 3) maxima and minima. After this sketch the graph of the function.

1. $y = 3 - 6x$
2. $y = px + q$ Assume that $p > 0$ and $q > 0$

3. $y = x - 3x^2$
4. $y = (a - x) \cdot (b - x) \cdot (c - x)$ Assume that $a, b, c > 0$ and $a < b < c$, namely $b = 2 \cdot a$ and $c = 3 \cdot a$
5. $y = x^3 - 5x^2$
6. $y = \frac{4x}{x+2} + 4$
7. $y = \frac{3x}{x+a} + 4$. Assume that $a > 0$ and only consider $x > 0$ and $y > 0$. Determine how the shape of the graph depends on the value of the parameter a.

Chapter 8

Answers

8.1 Exponentials and Logarithms

1. $7^{345} \times 7^{5} = 7^{350}$

2. $32^{20} \times 32^{18} = 32^{38}$

3. $2^{5} \times 2^{2} \times 2^{3} = 2^{10}$

4. $3^{25} \times 3^{25} \times 3^{25} = (3^{25})^{3} = 3^{75}$

5. In general the total number of bacteria after n generations is given by the formula:

   $$2^{generations} \times \text{initial number of bacteria = total number of bacteria after n generations}$$

   In 3 hours, B. cereus will divide 6 times. Therefore, $n = 6$. $2^{6} = 64$ and $100x64 = 6,400$ bacteria are expected.

6. We do not know the division time of the bacteria, but we can calculate it. We know that in two hours of time they increased 4 times in size, ie probably each bacteria made two cell divisions in two hours. Therefore the cell division time seems to be 1 hour. In the initial three hours each bacteria would undergo 3 rounds of cell division, ie if you start from a single bacteria in three hours you would reach: $2^{3} = 8$ bacteria. Given we have 10000 bacteria after 3 hours, the initial population should have been $10000/8 = 1250$.

7. $\frac{(n^{-3})^{4}}{(n^{4}p^{-3})^{-3}} = \frac{n^{-12}}{(n^{-12}p^{9})} = p^{-9}$.

8. $\log_{8} 512 = 3$ since $8^{3} = 512$.

9. if $\log(2x) = 3$ then $2x = 1000$ and $x = 500$.

10. If $5e^x = 11$, then $e^x = 11/5 = 2.2$. Taking the natural logarithm of both sides of the equation one gets: $x = \ln(2.2) = 0.788$

11. if $\log_2(64^x) = 36$, then $64^x = (2^{36})$, as $64 = 2^6$ we rewrite to $2^{6x} = 2^{36}$. Thus $x = 6$.

12. Taking the natural logarithm of both sides of the equation we get: $-ax = bx\ln(1/2)$. x cancels and we have: $-a = -b\ln(2)$, and $b = \frac{a}{\ln 2}$. Remember $\ln(1/2) = \ln(2^{-1}) = -\ln(2)$.

8.2 Fractions

1. (a) $\frac{1}{7} \times \frac{2}{4} = \frac{2}{28} = \frac{1}{14}$
   (b) $\frac{1}{6} - \frac{2}{3} = \frac{3}{18} - \frac{12}{18} = -\frac{9}{18} = -\frac{1}{2}$
   (c) $\frac{2}{3} : \frac{4}{5} = \frac{2}{3} \times \frac{5}{4} = \frac{10}{12} = \frac{5}{6}$
   (d) $\frac{4}{12} + \frac{5}{36} = \frac{12}{36} + \frac{5}{36} = \frac{17}{36}$
2. $2400 - (\frac{1}{3} + \frac{1}{7} + \frac{1}{4})2400 + \frac{1}{3}\frac{1}{3}2400 = 2400 - (\frac{1*7*4}{3*7*4} + \frac{1*3*4}{7*3*4} + \frac{1*3*7}{4*3*7})2400 + \frac{1}{9}2400 = 2400 - (\frac{28}{84} + \frac{12}{84} + \frac{21}{84})2400 + \frac{1}{9}2400 = 2400 - \frac{61}{84}2400 + \frac{2400}{9} = 2400 - 1742,85 + 266,67 = 2400 - 1476,18 = 923,81$
3. The abc-formula is in the form $ax^2 + bx + c = 0$. In this case instead of x, we have r. We need to first get all the r's to one side:

   $\frac{6}{r} = \frac{5r}{30r+5} \rightarrow \frac{6}{r} - \frac{5r}{30r+5} = 0 \rightarrow \frac{6}{r} - \frac{r}{6r+1} = 0$

   Now, we can subtract the two resulting fractions:

   $\frac{36r+6}{r(6r+1)} - \frac{r^2}{r(6r+1)} = 0 \rightarrow \frac{-r^2+36r+6}{r(6r+1)} = 0$

   Note that we now only need to worry about the numerator ($-r^2 + 36r + 6$), as this equation will only ever be 0 if the numerator is 0. So:

   $-r^2 + 36r + 6 = 0$

   Which can then be solved with the abc-formula.
4. $\frac{a}{b}x - \frac{c}{d} = \frac{e}{f}$ so $\frac{a}{b}x = \frac{e}{f} + \frac{c}{d}$ so $\frac{a}{b}x = \frac{ed+cf}{df}$ so $x = \frac{(ed+cf)b}{afd}$
5. $25A : \frac{3}{7} = 12 - \frac{1}{3}A$ so $25A\frac{7}{3} = 12 - \frac{1}{3}A$ so $\frac{175}{3}A = 12 - \frac{1}{3}A$ so $\frac{176}{3}A = 12$ so $A = 12\frac{3}{176} = \frac{36}{176} = \frac{18}{88} = \frac{9}{44}$
6. (a) $(b - \frac{N}{k})N = 0$ so $N = 0$ or $b - \frac{N}{k} = 0$ so $\frac{N}{k} = b$ so $N = bk$
   (b) $(b - d(1 + \frac{N}{k}))N = 0$ so $N = 0$ or $b - d(1 + \frac{N}{k}) = 0$ so $d + \frac{d}{k}N = b$ so $\frac{d}{k}N = b - d$ so $N = \frac{(b-d)k}{d}$
   (c) $(\frac{b}{1+N/h} - d)N = 0$ so $N = 0$ or $\frac{b}{1+N/h} - d = 0$ so $\frac{b}{1+N/h} = d$ so $b = d(1 + N/h)$ so $b = d + (d/h)N$ so $b - d = (d/h)N$ so $N = \frac{(b-d)h}{d}$

8.3 Algebraic equations

1. (a) $3axy - 4abx^2 - 6by^2 + 8b^2xy + 4abx^2 + 6by^2 - 8xyb^2 = 3axy$
   (b) $\frac{6(30r+5)}{r(30r+5)} - \frac{5r^2}{r(30r+5)} = \frac{6(6r+1)}{r(6r+1)} - \frac{r^2}{r(6r+1)} = \frac{-r^2+36r+6}{r(6r+1)}$
2. (a) $3r + 2 - 5r - 5 = 6r + 4$; so $-2r - 3 = 6r + 4$; so $-7 = 8r$; so $r = -\frac{7}{8}$

(b) $x+\frac{4}{x}=4$; so $x^2+4=4x$ and $x\neq 0$; so $x^2-4x+4=0$; so $x_{1,2}=\frac{4\pm\sqrt{4^2-4*1*4}}{2}$ which gives us $x_{1,2}=2\pm\sqrt{0}=2$

(c) $(b-\frac{N}{k})N=0$; so $N=0$, or $N=bk$.

(d) $N=0$, or $b-d(1+\frac{N}{k})=0$; so $b=-d(1+\frac{N}{k})$, $\frac{b}{d}=(1+\frac{N}{k})$ $\frac{b}{d}-1=\frac{N}{k}$ thus $N=k(\frac{b}{d}-1)$ or $N=\frac{k(b-d)}{d}$.

(e) $N=0$, or $\frac{b}{1+N/h}-d=0$; so $\frac{b}{1+N/h}=d$; so $b=d(1+N/h)$; so $b=d+(d/h)N$; $b-d=(d/h)N$, and thus $N=\frac{(b-d)h}{d}$ or $N=h(\frac{b}{d}-1)$

3. (a) From 1st eq. $x=2y-5$, substitution into 2nd eq. gives $2(2y-5)+y=10$ so $5y-10=10$, $5y=20$ so $y=4$, and thus $x=2*4-5=8-5=3$.

(b) From 1st eq. $x=-\frac{b}{a}y$ substitution into 2nd eq. gives $\frac{-cb}{a}y+dy=-b$ so $(d-\frac{cb}{a}y=-b$ or $\frac{da-cb}{a}y=-b$ so $y=\frac{-ba}{da-cb}$ and hence $x=-\frac{b}{a}y=-\frac{b}{a}\frac{-ba}{da-bc}=\frac{b^2}{da-bc}$

(c) From 2nd eq. $y(4-x)=0$, this gives $y=0$ or $x=4$. Substitution of $y=0$ into the 1st eq. gives $x(1-2x)=0$ and hence $x=0$ and $x=0.5$. Substitution of $x=4$ into the 1st eq. of gives $4(1-2*4)+4y=0$ so $4*-7+4y=0$ $-28=-4y$ so $y=7$. Thus the solutions are: $(0,0),(0.5,0),(4,7)$.

(d) From 2nd eq. $y(9-3x-y)=0$, we find $y=0$ or $9-3x-y=0$ so $y=9-3x$. Substitution of $y=0$ into the 1st eq. gives $4x-x^2=0$ or $x(4-x)=0$ so $x=0$ or $x=4$. Substitution of $y=9-3x$ into the 1st eq. gives $4x-x(9-3x)-x^2=0$ so $4x-9x+3x^2-x^2=0$ so $-5x+2x^2=0$ so $x(-5+2x)=0$ so $x=0$ or $x=-2.5$ and thus $y=9$ or $y=9-7.5=1.5$ Thus the solutions are: $(0,0),(4,0),(0,9),(2.5,1.5)$.

(e) From 2nd eq. $N=0$, or $R=\delta$. Substitution of $N=0$ into the 1st eq. gives $b(1-(R/k)-d)R=0$ so $R=0$ or $b(1-(R/k)-d)=0$ so $1-(R/k)-d=0$ $1-d=R/k$ so $R=k(1-d)$. Substitution of $R=\delta$ into the 1st eq. gives $b(1-(\delta/k)-d-aN)\delta=0$ so $1-(\delta/k)-d-aN=0$ so $1-(\delta/k)-d=aN$ so $N=\frac{1-d}{a}-\frac{\delta}{ak}$ Thus the solutions are: $(0,0),(k(1-d),0),(\delta,\frac{1-d}{a}-\frac{\delta}{ak})$.

8.4 Limits and Asymptots

1. The term $3x^2$ goes to $+\infty$ and the term $-2x$ to $-\infty$, but since the first contains a higher power of x it wins and the whole function will go to $+\infty$
2. x approaches zero from the positive side, that is you divide by very small positive numbers, resulting in very large positive numbers, so the function goes to $+\infty$
3. x approaches zero from the negative side, that is you divide by very small negative numbers, resulting in very large negative numbes, so the function goes to $-\infty$
4. If you would fill in $x=-1$ both the numerator and denominator become zero. Therefore, to find the limit we need to take the derivative of both the numerator ($f(x)=2+2x$ so $f'(x)=2$) and the denominator ($g(x)=1+x$ so $g'(x)=1$) (rule of l'Hopital) and this results in $\frac{2}{1}=2$.
5. To find the limit here we find the largest power of x and divide all terms in the numerator and denominator by that term. The largest power is here x^3. Dividing all terms by it results in $\frac{1/x+3/x^3}{1}$. Both $1/x$ and $3/x^3$ will go to zero for x going to ∞ thus the whole function will also go to 0.
6. Again, we need to determine the largest power of x and divide all terms by it. The largest power is now x^2, dividing all terms by it results in $\frac{a}{b+1/x}$. For x going to ∞ $1/x$ approaches zero so we are left with $\frac{a}{b}$

7. Largest power present in the fraction here is x^3, dividing all terms **in the fraction** by it results in $\frac{a/x+b/x^3}{c}+d$. Note that **you should not divide the separate term d by** x^3! (Dividing numerator and denominator by the same, means that the overall value of the fraction remains the same. Dividing a single number by some factor results in an entirely different value.) For x going to ∞ both a/x and b/x^3 go to zero, so the whole fraction goes to zero, and hence the function as a whole goes to d (note that this term is fully independent of what x does).

8. $\lim_{x\to\pm\infty}\frac{x}{x-2}+1=2$ so HA is $y=2$; for $x=2$ we get a division by zero, so $x=2$ is a VA, with $lim_{x\to 2+}\frac{x}{x-2}+1=+\infty$ and $lim_{x\to 2-}\frac{x}{x-2}+1=-\infty$. Solving for $f(x)=0$ gives $\frac{x}{x-2}+1=0$ so $\frac{x}{x-2}=-1$ so$x=-x+2$ so $2x=2$ so $x=1$ so intersection point with x axis is $(1,0)$. For $x=0$ $f(0)=1$ so intersection point with the y axis is $(0,1)$. Together this allows you to draw the function.

9. $\lim_{x\to\pm\infty}\frac{ax}{bx+c}-d=(a/b)-d$ (¡0) so HA is $y=(a/b)-d$; for $x=-(c/b)$ we get a division by zero, so $x=-(c/b)$ is a VA, with with $lim_{x\to-(c/b)+}\frac{ax}{bx+c}-d=-\infty$ and $lim_{x\to-(c/b)-}\frac{ax}{bx+c}-d=-\infty$. Solving for $f(x)=0$ gives $\frac{ax}{bx+c}-d=0$ so $\frac{ax}{bx+c}=d$ so $ax=bdx+cd$ so $(a-bd)x=cd$ so $x=cd/(a-bd)$ so intersection piont with x axis is $(cd/(a-bd),0)$ with $(cd/(a-bd))<0$. For $x=0$ $f(0)=-d$ so intersection point with the y axis is $(-d,0)$. Together this allows you to draw the function.

8.5 Derivatives

1. (a) $f'(x)=2x+15$
   (b) $f'(x)=5*12x^4=60x^4$
   (c) $f'(x)=-4*7x^{-5}=-\frac{28}{x^5}$
   (d) $f'(x)=3*ax^2-5*bx^4$
2. (a) $f'(x)=3e^{5x^2}*2*5x=30xe^{5x^2}$ (chainrule)
   (b) $f'(x)=3*12x^2*6x^4+12x^3*4*6x^3=216x^6+288x^6=504x^6$ (productrule)
   (c) $f'(x)=-sin(ax-bx^3)*(a-3*bx^2)=(-a+3*bx^2)sin(ax-bx^3)$ (chainrule)
   (d) $f'(x)=\frac{(5+2*6x)2x-(5x+6x^2)2}{(2x)^2}=\frac{10x+24x^2-10x-12x^2}{4x^2}=\frac{12x^2}{4x^2}=3$ (quotientrule)
3. (a) $f'(x)=\frac{3}{x}3*4x^2=\frac{36x^2}{x}=9/x$ (chainrule)
   (b) $f'(x)=cos(ax)*a*cos(ax)+sin(ax)*-sin(ax)*a=a*cos^2(ax)-a*sin^2(ax)$ (productrule)
   (c) $f'(x)=\frac{2*3x*6x-3x^2*6}{(6x)^2}=\frac{36x^2-18x^2}{36x^2}=\frac{18x^2}{36x^2}=0.5$ (quotientrule)
   (d) $f'(x)=2sin(x)cos(x)+2cos(x)-sin(x)=2sin(x)cos(x)-2sin(x)cos(x)=0$ (chainrule)
   (e) One option would be to apply the productrule twice, but in this case it is easer to first remove the brackets, rewriting the function as $f(x)=x^3-(a+c-b)x^2+(ac-bc-ab)x+abc$, then the derivative can be found very simply as $f'(x)=3x^2-2*(a+c-b)x+(ac-bc-ab)$

8.6 Drawing Functions

1. intersection point x axis: $y=0$: $3-6x=0$ so $x=0.5$
   intersection point y axis: $x=0$: $y=3-6*0=3$
   $\lim_{x\to+\infty}3-6x=-\infty$ and $\lim_{x\to-\infty}3-6x=+\infty$ so no HA
   no fraction so also no VA

$f(x) = 3 - 6x$ so $f'(x) = 3$ is never equal to zero so no maxima/minima
descending straight line through points (0,3) and (0.5,0)

2. intersection point x axis: $y = 0$: $px + q = 0$ so $x = -q/p$, given that $p > 0$ and $q > 0$ we know that $-q/p < 0$. intersection point y axis: $x = 0$: $y = p * 0 + q = q$, and $q > 0$ $\lim_{x\to+\infty} px + q = +\infty$ and $\lim_{x\to-\infty} px + q = -\infty$ so no HA
no fraction so also no VA
$f(x) = px + q$ so $f'(x) = p$ is never equal to zero so no maxima/minima
ascending straight line through points (0,-(q/p)) and (q,0)

3. intersection point x axis: $y = 0$: $x - 3x^2 = 0$ so $x(1 - 3x) = 0$ so $x = 0$ or $x = 1/3$
intersection point y axis: $x = 0$: $y = 0$
$\lim_{x\to+\infty} x - 3x^2 = -\infty$ and $\lim_{x\to-\infty} x - 3x^2 = -\infty$ so no HA (already allows you to see that it is a parabola with a maximum (*bergparabool*))
no fraction so no VA
$f(x) = x - 3x^2$ so $f'(x) = 1 - 6x$ $1 - 6x = 0$ so zero for $x = 1/6$
$f(x) = x - 3x^2$ is a parabola, since $a = -3$ it has a maximum (*bergparabool* intersection points with x axis are (0.0) en (1/3,0) and maximum lies at (1/6,1/12)

4. Note that for finding intersection points with the x-axis it is easier to *not* remove the brackets:
intersection points x axis: $y = 0$ if $a - x = 0$ or $b - x = 0$ or $c - x = 0$, so if $x = a$ or $x = b$ or $x = c$.
intersection points y axis: $x = 0$: $y = abc$, since $a, b, c > 0$ $abc > 0$, this already allows us to determine that this is a cubic, third degree function which comes from $+\infty$ for negative values of x and goes to $-\infty$ for positive values of x
indeed $\lim_{x\to+\infty}(a - x) * (b - x) * (c - x) = -\infty$ and $\lim_{x\to-\infty}(a - x) * (b - x) * (c - x) = +\infty$, which also means that there is no HA.
there is no fraction so there is also no VA (both these facts also follow from the function being a cubic function)
For finding maxima and minima we need the derivative of the function and for finding the derivative it actually is easier to remove the brackets:
$f(x) = (a - x) * (b - x) * (c - x) = (x^2 - (a + b)x + ab) * (c - x) = -x^3 + (a + b + c)x^2 - (ac + ab + bc)x + abc$ so $f'(x) = -3x^2 + 2(a + b + c)x - (ac + ab + bc)$. We can see that equating this to zero and solving it will become quite messy. However we know that $b = 2 * a$ and $c = 3 * a$ which can help us make it slightly less messy. $f'(x) = -3x^2 + 2(a + b + c)x - (ac + ab + bc)$ can be rewritten into $f'(x) = -3x^2 + 2 * (a + 2a + 3a) - (a * 3a + a * 2a + 2a * 3a) = -3x^2 + 12a + 11a^2$. Now we can solve $f'(x) = -3x^2 + 12a + 11a^2 = 0$ using the abc-formula: $x = \frac{-12a \pm \sqrt{(12a)^2 - 4 * -3 * 11a^2}}{2 * -3} = \frac{-12a \pm \sqrt{144a^2 - 132a^2}}{-6} = \frac{-12a \pm \sqrt{12a^2}}{-6} = \frac{-12a \pm a\sqrt{4 * 3}}{-6} = \frac{-12a \pm 2a\sqrt{3}}{-6} = 2a \mp \frac{2a\sqrt{3}}{6} = 2a \mp \frac{a\sqrt{3}}{3} = 2a \mp \frac{a}{\sqrt{3}}$.
Logically, $x = 2a - \frac{a}{\sqrt{3}}$ lies between $x = a$ and $x = b$ and represents a minimum, whereas $x = 2a + \frac{a}{\sqrt{3}}$ lies between $x = b$ and $x = c$ and represents a maximum (considering the shape of the function).

5. intersection point x axis: $y = 0$: $x^3 - 5x^2 = 0$ dus $x^2(x - 5) = 0$ so $x = 0$ if $x = 5$
intersection point y axis: $x = 0$: $y = 0$
$\lim_{x\to+\infty} x^3 - 5x^2 = +\infty$ and $\lim_{x\to-\infty} x^3 - 5x^2 = -\infty$ so no HA (already allows you to determine that this third degree function runs from the upperleft to the lowerright)
no fraction so no VA
$f(x) = x^3 - 5x^2$ so $f'(x) = 3x^2 - 10x$ maxima/minima for $3x^2 - 10x = 0$ can be written as $x(3x - 10) = 0$ and hence $x = 0$ and $x = 10/3$
Note that $x = 0$ is both an intersection point with the x axis and a maximum/minimum so at the

point (0,0) the function touches the x axis rather than truely intersecting it
$f(x) = x^3 - 5x^2$ from the left it comes from $+\infty$, touches the x axis (0,0) which hence is a minimum, after this the function goes up again so $x = 10/3$ is a maximum, and after this the function decreases again to $-\infty$ on its way intersecting the x axis at (5,0)

6. intersection point x axis: $y = \frac{4x}{x+2} + 4 = 0$ so $\frac{4x}{x+2} = -4$ so $4x = -4x - 8$ so $8x = -8$ and hence $x = -1$
intersection point y axis: $y = \frac{4*0}{0+2} + 4 = 4$
$\lim_{x\to\pm\infty} \frac{4x}{x+2} + 4 = 4 + 4 = 8$ so HA $y = 8$
fraction of which denominator can become zero for $x = -2$ so VA:
$\lim_{x\to-2^+} \frac{4x}{x+2} + 4 = -\infty$ $\lim_{x\to^-2^-} \frac{4x}{x+2} + 4 = +\infty$
$f(x) = \frac{4x}{x+2} + 4$ so $f'(x) = \frac{8}{(x+2)^2}$ can never be zero so no maxima/minima
$f(x) = \frac{4x}{x+2} + 4$ has HA at $y = 8$ and VA at $x = -2$, and passes through points (-1,0) and (0,4), left from $x = -2$ it goes to $+\infty$ while right from $x = -2$ it goes to $-\infty$

7. Note that this function is similar to the previous one!
intersection point x axis: $y = \frac{3x}{x+a} + 4 = 0$ so $\frac{3x}{x+a} = -4$ so $3x = -4x - 4a$ so $7x = -4a$ so $x = -4a/7$
intersection point y axis: $y = \frac{4*0}{0+a} + 4 = 4$
$\lim_{x\to\pm\infty} \frac{3x}{x+a} + 4 = 3 + 4 = 7$ so HA $y = 7$
fraction of which denominator can become zero for $x = -a$ so VA:
$\lim_{x\to-a^+} \frac{3x}{x+a} + 4 = -\infty$ $\lim_{x\to-a^-} \frac{3x}{x+a} + 4 = +\infty$
$f(x) = \frac{3x}{x+a} + 4$ so $f'(x) = \frac{3a}{(x+a)^2}$ can never be zero so no maxima/minima
$f(x) = \frac{3x}{x+a} + 4$ has HA at $y = 7$ and VA at $x = -a$, and passes through points (-4a/7,0) and (0,4), left from $x = -a$ it goes to $+\infty$ while right from $x = -a$ it goes to $-\infty$
Dependence on a:
You can find this dependence by reasoning as follows: the smaller a is, the faster a can be neglected relative to x and hence the faster the function approaches the asymptot $y = 7$ and hence the steeper the slope
You can also find this from the derivative: $f'(x) = \frac{3a}{(x+a)^2} = \frac{3a}{x^2+2ax+a^2}$ for the same x value a smaller a implies a steeper slope